Ch22
Linear Complexity
“Sweeping” Scheme

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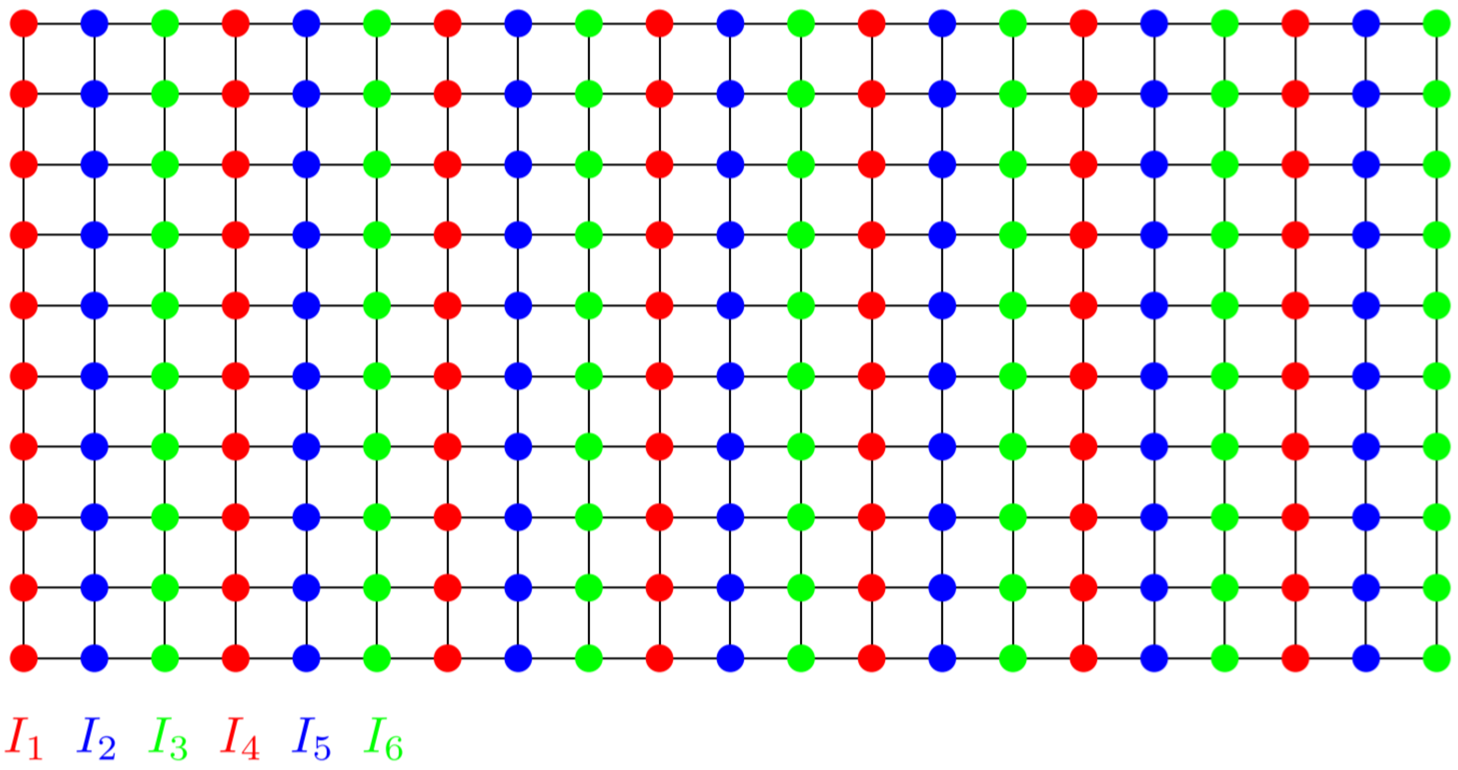
WARNING!

The algorithm is….

- Neither the most efficient one
- Nor the most general one that exists!
- But simple both to describe and to implement when using uniform gird!

Ch22.1 A basic sweeping scheme (uniform rectangular grids)

Partition of nodes



$$I = I_1 \cup I_2 \cup \cdots \cup I_{n_1},$$

The grid:

n_1 columns, n_2 rows

Each I_k holds the points in the kth column of the grid

Linear system

$$
\begin{bmatrix}
\mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\
\mathbf{A}_{2,1} & \mathbf{A}_{2,2} & \mathbf{A}_{2,3} & \mathbf{0} & \mathbf{0} & \cdots \\
\mathbf{0} & \mathbf{A}_{3,2} & \mathbf{A}_{3,3} & \mathbf{A}_{3,4} & \mathbf{0} & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{A}_{4,3} & \mathbf{A}_{4,4} & \mathbf{A}_{4,5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \\ \vdots
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \vdots
\end{bmatrix}.
$$

Notice that

- Points in I_k only interact with I_{k-1} and I_{k+1} !
- Each $A_{k,j}$ is also banded!

Linear system

$$\begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} & \mathbf{A}_{2,3} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{A}_{3,2} & \mathbf{A}_{3,3} & \mathbf{A}_{3,4} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{4,3} & \mathbf{A}_{4,4} & \mathbf{A}_{4,5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \vdots \end{bmatrix}.$$

Schur complement (i)

$$\mathbf{A}_{1,1}\mathbf{u}_1 + \mathbf{A}_{1,2}\mathbf{u}_2 = \mathbf{f}_1,$$

$$\mathbf{A}_{2,1}\mathbf{u}_1 + \mathbf{A}_{2,2}\mathbf{u}_2 + \mathbf{A}_{2,3}\mathbf{u}_3 = \mathbf{f}_2.$$

$$\Rightarrow \quad \left(\mathbf{A}_{2,2} - \mathbf{A}_{2,1}\mathbf{A}_{1,1}^{-1}\mathbf{A}_{1,2}\right)\mathbf{u}_2 + \mathbf{A}_{2,3}\mathbf{u}_3 = \mathbf{f}_2 - \mathbf{A}_{2,1}\mathbf{A}_{1,1}^{-1}\mathbf{f}_1.$$

Schur complement (ii)

$$\mathbf{A}_{1,1}\mathbf{u}_1 + \mathbf{A}_{1,2}\mathbf{u}_2 = \mathbf{f}_1,$$

$$\mathbf{A}_{2,1}\mathbf{u}_1 + \mathbf{A}_{2,2}\mathbf{u}_2 + \mathbf{A}_{2,3}\mathbf{u}_3 = \mathbf{f}_2.$$

$$\Rightarrow \quad \left(\mathbf{A}_{2,2} - \mathbf{A}_{2,1}\mathbf{A}_{1,1}^{-1}\mathbf{A}_{1,2}\right)\mathbf{u}_2 + \mathbf{A}_{2,3}\mathbf{u}_3 = \mathbf{f}_2 - \mathbf{A}_{2,1}\mathbf{A}_{1,1}^{-1}\mathbf{f}_1.$$

$$\Rightarrow \quad \mathbf{S}_2\mathbf{u}_2 + \mathbf{A}_{2,3}\mathbf{u}_3 = \tilde{\mathbf{f}}_2.$$

With $\mathbf{S}_2 = \mathbf{A}_{2,2} - \mathbf{A}_{2,1}\mathbf{A}_{1,1}^{-1}\mathbf{A}_{1,2}$, $\tilde{\mathbf{f}}_2 = \mathbf{f}_2 - \mathbf{A}_{2,1}\mathbf{A}_{1,1}^{-1}\mathbf{f}_1,$

Schur complement (iii)

$$\mathbf{S}_2\mathbf{u}_2 + \mathbf{A}_{2,3}\mathbf{u}_3 = \tilde{\mathbf{f}}_2.$$

$$\mathbf{A}_{3,2}\mathbf{u}_2 + \mathbf{A}_{3,3}\mathbf{u}_3 + \mathbf{A}_{3,4}\mathbf{u}_4 = \mathbf{f}_3.$$

Keep going...

$$\Rightarrow \quad \left(\mathbf{A}_{3,3} - \mathbf{A}_{3,2}\mathbf{S}_2^{-1}\mathbf{A}_{2,3}\right)\mathbf{u}_3 + \mathbf{A}_{3,4}\mathbf{u}_4 = \mathbf{f}_3 - \mathbf{A}_{3,2}\mathbf{S}_2^{-1}\tilde{\mathbf{f}}_2.$$

$$\Rightarrow \quad \mathbf{S}_3\mathbf{u}_3 + \mathbf{A}_{3,4}\mathbf{u}_4 = \tilde{\mathbf{f}}_3.$$

With $\mathbf{S}_3 = \mathbf{A}_{3,3} - \mathbf{A}_{3,2}\mathbf{S}_2^{-1}\mathbf{A}_{2,3}$, $\tilde{\mathbf{f}}_3 = \mathbf{f}_3 - \mathbf{A}_{3,2}\mathbf{S}_2^{-1}\tilde{\mathbf{f}}_2,$

Schur complement (iv)

$$\mathbf{S}_{n_1}\mathbf{u}_{n_1} = \tilde{\mathbf{f}}_{n_1},$$

End of 'rightward' sweeping

$$\Rightarrow \quad \mathbf{u}_{n_1} = \mathbf{S}_{n_1}^{-1}\tilde{\mathbf{f}}_{n_1}.$$

Start of 'leftward' sweeping

Schur complement (v)

Sequentially, we have

$$\mathbf{S}_{n_1}\mathbf{u}_{n_1} = \tilde{\mathbf{f}}_{n_1}, \quad \Longrightarrow \quad \mathbf{u}_{n_1} = \mathbf{S}_{n_1}^{-1}\tilde{\mathbf{f}}_{n_1}.$$

$$\mathbf{S}_{n_1-1}\mathbf{u}_{n_1-1} + \mathbf{A}_{n_1-1,n_1}\mathbf{u}_{n_1} = \tilde{\mathbf{f}}_{n_1-1}, \quad \Longrightarrow \quad \mathbf{u}_{n_1-1} = \mathbf{S}_{n_1-1}^{-1}\left(\tilde{\mathbf{f}}_{n_1-1} - \mathbf{A}_{n_1-1,n_1}\mathbf{u}_{n_1}\right).$$

$$\vdots \qquad\qquad \Downarrow$$

$$\mathbf{A}_{1,1}\mathbf{u}_1 + \mathbf{A}_{1,2}\mathbf{u}_2 = \mathbf{f}_1, \quad \Longrightarrow \quad \mathbf{u}_1 = \mathbf{A}_{1,1}^{-1}\left(\mathbf{f}_1 - \mathbf{A}_{1,2}\mathbf{u}_2\right).$$

Schur complement (vi)

The iteration of Schur Complements

(Actually we care about its inverse!)

Define $\mathbf{X}_k = \mathbf{S}_k^{-1}$

$$\mathbf{X}_k = \begin{cases} \mathbf{A}_{1,1}^{-1}, & \text{when } k = 1, \\ \left(\mathbf{A}_{k,k} - \mathbf{A}_{k,k-1}\mathbf{X}_{k-1}\mathbf{A}_{k-1,k}\right)^{-1}, & \text{when } k > 1. \end{cases}$$

Algorithm

Complexity:
$O(n_1 n_2) = O(N)$
Linear!

INVERTING AND SOLVING A BLOCK-TRIDIAGONAL LINEAR SYSTEM

Build all solution operators in a rightward sweep:

$\mathbf{X}_1 = \mathbf{A}(I_1, I_1)^{-1}$

for $i = 2 : m$

$\quad \mathbf{X}_i = \left(\mathbf{A}(I_i, I_i) - \mathbf{A}_{i,i-1}\mathbf{X}_{i-1}\mathbf{A}_{i-1,i}\right)^{-1}$

end for

Given a load $\mathbf{f}$, *compute equivalent loads in a rightward sweep:*

$\tilde{\mathbf{f}}(I_1) = \mathbf{f}(I_1)$

for $i = 2 : m$

$\quad \tilde{\mathbf{f}}_i = \mathbf{f}(I_i) - \mathbf{A}(I_i, I_{i-1})\,\mathbf{X}_{i-1}\tilde{\mathbf{f}}(I_{i-1})$

end for

Given the equivalent loads, compute solutions in a leftward sweep:

$\mathbf{u}(I_m) = \mathbf{X}_m\,\tilde{\mathbf{f}}(I_m)$

for $i = (m-1) : (-1) : 1$

$\quad \mathbf{u}(I_i) = \mathbf{X}_i\left(\tilde{\mathbf{f}}(I_i) - \mathbf{A}_{i,i+1}\mathbf{u}(I_{i+1})\right)$

end for

Important Message

The algorithm right now…

- Nothing more than traditional Gaussian elimination with column-by-column ordering
- Can be accelerated since these Schur complements are highly compressible!

Remarks

- An alternative solver is based on LU factorization

$$\mathbf{A} = \mathbf{L}\mathbf{U} = \begin{bmatrix} \mathbf{L}_{1,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{L}_{2,1} & \mathbf{L}_{2,2} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{L}_{3,2} & \mathbf{L}_{3,3} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{L}_{4,3} & \mathbf{L}_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \mathbf{U}_{1,1} & \mathbf{U}_{1,2} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{U}_{2,2} & \mathbf{U}_{2,3} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{U}_{3,3} & \mathbf{U}_{3,4} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{U}_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Remarks

It leads to the recursive formula:

$$[\mathbf{L}_{1,1},\ \mathbf{U}_{1,1}] = \mathtt{lu}(\mathbf{A}_{1,1}).$$

$$\mathbf{L}_{i,i-1} = \mathbf{A}_{i,i-1}\mathbf{U}_{i-1,i-1}^{-1},$$

$$\mathbf{U}_{i-1,i} = \mathbf{L}_{i-1,i-1}^{-1}\mathbf{A}_{i-1,i},$$

$$[\mathbf{L}_{i,i},\ \mathbf{U}_{i,i}] = \mathtt{lu}(\mathbf{A}_{i,i} - \mathbf{L}_{i,i-1}\mathbf{U}_{i-1,i}).$$

PS: We can always use a Cholesky factorization when A is SPD!

Algorithm (LU)

LU FACTORIZATION AND SOLVE OF A BLOCK-TRIDIAGONAL SYSTEM

Build the LU factorization through a rightward sweep:

$[\mathbf{L}_{1,1}, \mathbf{U}_{1,1}] = \texttt{lu}(\mathbf{A}_{1,1})$

for $i = 2 : m$

$\quad \mathbf{S}_i = \mathbf{A}_{i,i} - \mathbf{A}_{i,i-1}\mathbf{U}_{i-1,i-1}^{-1}\mathbf{L}_{i-1,i-1}^{-1}\mathbf{A}_{i-1,i}$

$\quad [\mathbf{L}_{i,i}, \mathbf{U}_{i,i}] = \texttt{lu}(\mathbf{S}_i)$

end for

Given a load $\mathbf{f}$*, solve* $\mathbf{L}\mathbf{y} = \mathbf{f}$ *in a rightward sweep:*

$\mathbf{y}(I_1) = \mathbf{L}_{1,1}^{-1}\,\mathbf{f}(I_1)$

for $i = 2 : m$

$\quad \mathbf{y}(I_i) = \mathbf{L}_{i,i}^{-1}\left(\mathbf{f}(I_i) - \mathbf{A}_{i,i-1}\,\mathbf{U}_{i-1,i-1}^{-1}\mathbf{u}(I_{i-1})\right)$

end for

Given $\mathbf{y}$*, solve* $\mathbf{U}\mathbf{u} = \mathbf{y}$ *in a leftward sweep:*

$\mathbf{u}(I_m) = \mathbf{U}_{m,m}^{-1}\,\mathbf{y}(I_m)$

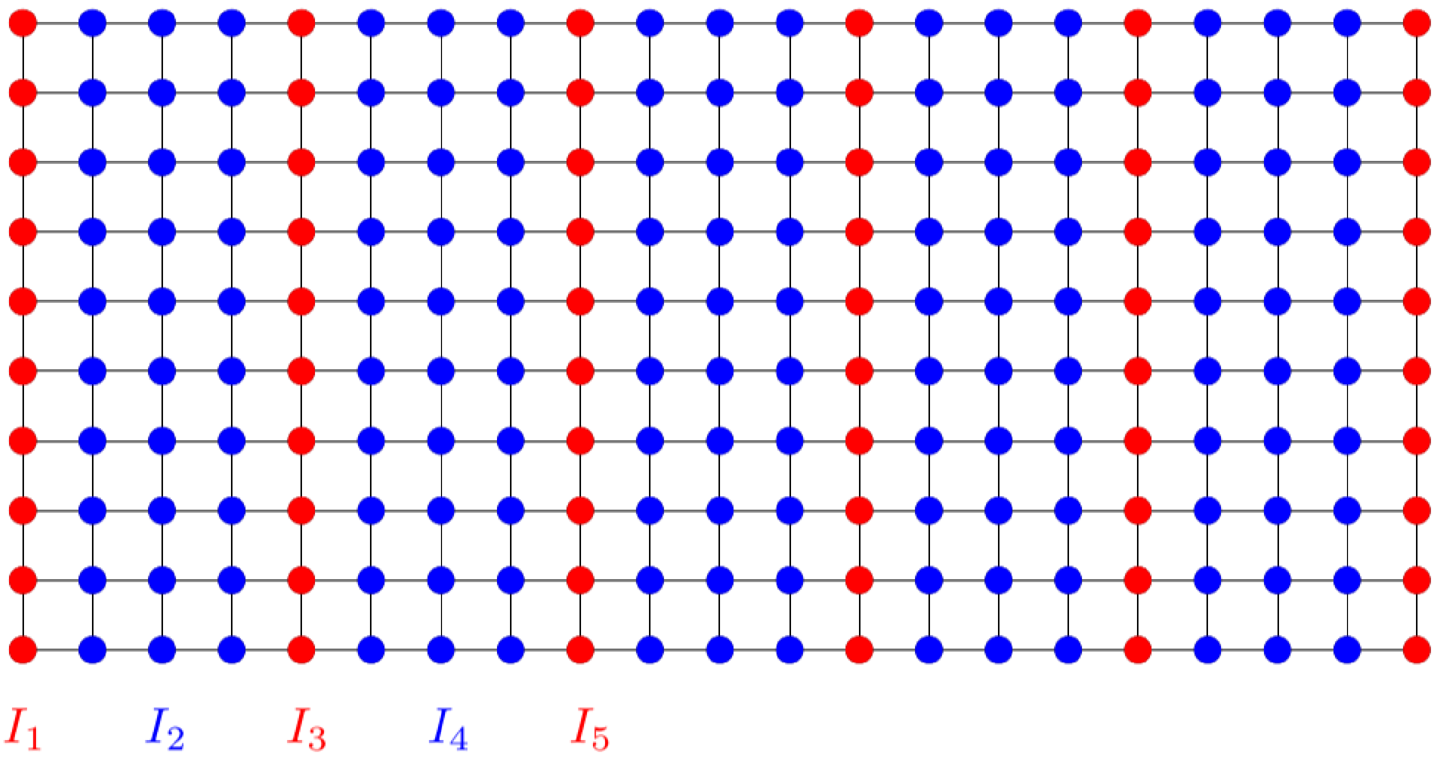
for $i = (m-1) : (-1) : 1$

$\quad \mathbf{u}(I_i) = \mathbf{U}_{i,i}^{-1}\left(\mathbf{y}(I_i) - \mathbf{L}_{i,i}^{-1}\mathbf{A}_{i,i+1}\mathbf{u}(I_{i+1})\right)$

end for

Ch22.2 Buffered sweeping schemes

Partition of nodes (Buffered)



$$I = I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5 \cup \cdots \cup I_{2m+1}.$$

The grid:
n_1 columns, n_2 rows

Buffers: $I_2, I_4, I_6, \ldots.$
Each holds b
columns of the grid

$$n_1 = 1 + m(b+1)$$

Linear system

$$\begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} & \mathbf{A}_{2,3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{A}_{3,2} & \mathbf{A}_{3,3} & \mathbf{A}_{3,4} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{4,3} & \mathbf{A}_{4,4} & \mathbf{A}_{4,5} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{5,4} & \mathbf{A}_{5,5} & \mathbf{A}_{5,6} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{6,5} & \mathbf{A}_{6,6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \\ \mathbf{u}_5 \\ \mathbf{u}_6 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \\ \vdots \end{bmatrix}.$$

Linear system

$$\begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} & \mathbf{A}_{2,3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{A}_{3,2} & \mathbf{A}_{3,3} & \mathbf{A}_{3,4} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{4,3} & \mathbf{A}_{4,4} & \mathbf{A}_{4,5} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{5,4} & \mathbf{A}_{5,5} & \mathbf{A}_{5,6} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{6,5} & \mathbf{A}_{6,6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \\ \mathbf{u}_5 \\ \mathbf{u}_6 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \\ \vdots \end{bmatrix}.$$

Schur complement (i)

$$\mathbf{A}_{2,1}\mathbf{u}_1 + \mathbf{A}_{2,2}\mathbf{u}_2 + \mathbf{A}_{2,3}\mathbf{u}_3 = \mathbf{f}_2 .$$

$$\Rightarrow \quad \mathbf{u}_2 = \mathbf{A}_{2,2}^{-1}\big(\mathbf{f}_2 - \mathbf{A}_{2,1}\mathbf{u}_1 - \mathbf{A}_{2,3}\mathbf{u}_3\big) .$$

Schur complement (ii)

$$\begin{bmatrix} (\mathbf{A}_{1,1} - \mathbf{A}_{1,2}\mathbf{A}_{2,2}^{-1}\mathbf{A}_{2,1}) & -\mathbf{A}_{1,2}\mathbf{A}_{2,2}^{-1}\mathbf{A}_{2,3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ -\mathbf{A}_{3,2}\mathbf{A}_{2,2}^{-1}\mathbf{A}_{2,1} & (\mathbf{A}_{3,3} - \mathbf{A}_{3,2}\mathbf{A}_{2,2}^{-1}\mathbf{A}_{2,3}) & \mathbf{A}_{3,4} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{A}_{4,3} & \mathbf{A}_{4,4} & \mathbf{A}_{4,5} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{5,4} & \mathbf{A}_{5,5} & \mathbf{A}_{5,6} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{6,5} & \mathbf{A}_{6,6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \\ \mathbf{u}_5 \\ \mathbf{u}_6 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 - \mathbf{A}_{1,2}\mathbf{A}_{2,2}^{-1}\mathbf{f}_2 \\ \mathbf{f}_3 - \mathbf{A}_{3,2}\mathbf{A}_{2,2}^{-1}\mathbf{f}_2 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \\ \vdots \end{bmatrix}.$$

Schur complement (ii)

$$\begin{bmatrix} (\mathbf{A}_{1,1} - \mathbf{A}_{1,2}\mathbf{A}_{2,2}^{-1}\mathbf{A}_{2,1}) & -\mathbf{A}_{1,2}\mathbf{A}_{2,2}^{-1}\mathbf{A}_{2,3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ -\mathbf{A}_{3,2}\mathbf{A}_{2,2}^{-1}\mathbf{A}_{2,1} & (\mathbf{A}_{3,3} - \mathbf{A}_{3,2}\mathbf{A}_{2,2}^{-1}\mathbf{A}_{2,3}) & \mathbf{A}_{3,4} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{A}_{4,3} & \mathbf{A}_{4,4} & \mathbf{A}_{4,5} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{5,4} & \mathbf{A}_{5,5} & \mathbf{A}_{5,6} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{6,5} & \mathbf{A}_{6,6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \\ \mathbf{u}_5 \\ \mathbf{u}_6 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 - \mathbf{A}_{1,2}\mathbf{A}_{2,2}^{-1}\mathbf{f}_2 \\ \mathbf{f}_3 - \mathbf{A}_{3,2}\mathbf{A}_{2,2}^{-1}\mathbf{f}_2 \\ \mathbf{f}_4 \\ \mathbf{f}_5 \\ \mathbf{f}_6 \\ \vdots \end{bmatrix}.$$

Keep eliminating $u_4, u_6, \ldots$
in an analogous manner!

New Linear system (i)

$$
\begin{bmatrix}
\tilde{\mathbf{A}}_{1,1} & \tilde{\mathbf{A}}_{1,3} & \mathbf{0} & \mathbf{0} & \cdots \\
\tilde{\mathbf{A}}_{3,3} & \tilde{\mathbf{A}}_{3,3} & \tilde{\mathbf{A}}_{3,5} & \mathbf{0} & \cdots \\
\mathbf{0} & \tilde{\mathbf{A}}_{5,3} & \tilde{\mathbf{A}}_{5,5} & \tilde{\mathbf{A}}_{5,7} & \cdots \\
\mathbf{0} & \mathbf{0} & \tilde{\mathbf{A}}_{7,5} & \tilde{\mathbf{A}}_{7,7} & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_1 \\ \mathbf{u}_3 \\ \mathbf{u}_5 \\ \mathbf{u}_7 \\ \vdots
\end{bmatrix}
=
\begin{bmatrix}
\tilde{\mathbf{f}}_1 \\ \tilde{\mathbf{f}}_3 \\ \tilde{\mathbf{f}}_5 \\ \tilde{\mathbf{f}}_7 \\ \vdots
\end{bmatrix},
$$

New Linear system (ii)

Diagonal :

$$\tilde{\mathbf{A}}_{2k+1,2k+1} = \mathbf{A}_{2k+1,2k+1} - \mathbf{A}_{2k+1,2k}\mathbf{A}_{2k,2k}^{-1}\mathbf{A}_{2k,2k+1} - \mathbf{A}_{2k+1,2k+2}\mathbf{A}_{2k+2,2k+2}^{-1}\mathbf{A}_{2k+2,2k+1},$$

Off-Diagonal :

$$\tilde{\mathbf{A}}_{2k-1,2k+1} = -\mathbf{A}_{2k-1,2k}\mathbf{A}_{2k,2k}^{-1}\mathbf{A}_{2k,2k+1},$$

$$\tilde{\mathbf{A}}_{2k+1,2k-1} = -\mathbf{A}_{2k+1,2k}\mathbf{A}_{2k,2k}^{-1}\mathbf{A}_{2k,2k-1}.$$

And

$$\tilde{\mathbf{A}}_{1,1} = \mathbf{A}_{1,1} - \mathbf{A}_{1,2}\mathbf{A}_{2,2}^{-1}\mathbf{A}_{2,1},$$

$$\tilde{\mathbf{A}}_{2m+1,2m+1} = \mathbf{A}_{2m+1,2m+1} - \mathbf{A}_{2m+1,2m}\mathbf{A}_{2m,2m}^{-1}\mathbf{A}_{2m,2m+1}.$$

New Linear system (iii)

Given the new reduced system

$$\tilde{\mathbf{A}}\tilde{\mathbf{u}} = \tilde{\mathbf{f}}.$$

We use the algorithm in Chapter 22.1 to solve it!

Now it's much smaller and more practicable!

Algorithm (i)

A BUFFERED SWEEPING SCHEME

Initialize the block-tridiagonal matrix $\tilde{\mathbf{A}}$ by copying over the diagonal blocks from $\mathbf{A}$:

for $k = 1 : (m+1)$

$\tilde{\mathbf{A}}_{2k-1,2k-1} = \mathbf{A}_{2k-1,2k-1}$

end for

Eliminate all buffer nodes (the loop can be executed in any order):

for $k = 1 : m$

$\tilde{\mathbf{A}}_{2k-1,2k-1} = \tilde{\mathbf{A}}_{2k-1,2k-1} - \mathbf{A}_{2k-1,2k}\mathbf{A}_{2k,2k}^{-1}\mathbf{A}_{2k,2k-1}$

$\tilde{\mathbf{A}}_{2k+1,2k+1} = \tilde{\mathbf{A}}_{2k+1,2k+1} - \mathbf{A}_{2k+1,2k}\mathbf{A}_{2k,2k}^{-1}\mathbf{A}_{2k,2k+1}$

$\tilde{\mathbf{A}}_{2k-1,2k+1} = \mathbf{A}_{2k-1,2k}\mathbf{A}_{2k,2k}^{-1}\mathbf{A}_{2k,2k+1}$

$\tilde{\mathbf{A}}_{2k+1,2k-1} = \mathbf{A}_{2k+1,2k}\mathbf{A}_{2k,2k}^{-1}\mathbf{A}_{2k,2k-1}$

end for

Fully parallelizable!

Factorize the block-tridiagonal system in a rightward sweep:

$\mathbf{X}_1 = \tilde{\mathbf{A}}_{1,1}^{-1}$.

for $k = 1 : m$

$\mathbf{X}_{2k+1} = \left(\tilde{\mathbf{A}}_{2k+1,2k+1} - \tilde{\mathbf{A}}_{2k+1,2k-1}\mathbf{X}_{2k-1}\tilde{\mathbf{A}}_{2k-1,2k+1}^{-1}\right)^{-1}$

end for

Algorithm (ii)

Given a load $\mathbf{f}$*, compute equivalent loads for the block-tridiagonal system:*

for $k = 1 : (m+1)$

$\quad \tilde{\mathbf{f}}_{2k-1} = \mathbf{f}(I_{2k-1})$

end for

for $k = 1 : m$

$\quad \tilde{\mathbf{f}}_{2k-1} = \tilde{\mathbf{f}}_{2k-1} - \mathbf{A}_{2k-1,2k}\,\mathbf{A}_{2k,2k}^{-1}\mathbf{f}(I_{2k})$

$\quad \tilde{\mathbf{f}}_{2k+1} = \tilde{\mathbf{f}}_{2k+1} - \mathbf{A}_{2k+1,2k}\,\mathbf{A}_{2k,2k}^{-1}\mathbf{f}(I_{2k})$

end for

Given the equivalent loads, compute solutions on the interfaces in a rightward sweep:

$\mathbf{u}(I_{2m+1}) = \mathbf{X}_{2m+1}\,\tilde{\mathbf{f}}_{2m+1}$

for $k = m : (-1) : 1$

$\quad \mathbf{u}(I_{2k-1}) = \mathbf{X}_{2k-1}\big(\tilde{\mathbf{f}}_{2k-1} - \tilde{\mathbf{A}}_{2k-1,2k+1}\mathbf{u}(I_{2k+1})\big)$

end for

Solve for the solutions on all buffer nodes (the loop can be executed in any order):

for $k = 1 : m$

$\quad \mathbf{u}(I_{2k}) = \mathbf{A}_{2k,2k}^{-1}\big(\mathbf{f}(I_{2k}) - \mathbf{A}_{2k,2k-1}\mathbf{u}(I_{2k-1}) - \mathbf{A}_{2k,2k+1}\mathbf{u}(I_{2k+1})\big)$

end for

Implementation considerations

- Each matrix $\mathbf{A}_{2k,2k}$ for two-dimensional problems will be banded. $\mathbf{A}_{2k,2k}^{-1}$ can be applied very rapidly.
- Parallel implementation: Computing the block-tridiagonal matrix $\widetilde{\mathrm{A}}$ is trivially parallelizable since each buffer region can be eliminated independently.

Thank you!

Q&A time