Sep 2, 2020

Fast solvers : An introduction

The algorithms of the course will address solving <u>elliptic PDEs</u> : E.g. $\Delta u = f$ Poisson equation.

Other examples: Fluid flow : incompressible (Stokes flow)

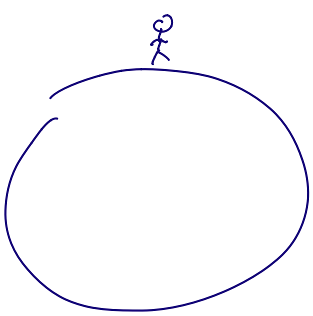
Electromagnetics : Maxwell's Equations

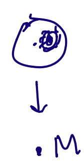
Elasticity ..

Example: Calculate the gravitational potential due to the moon: $\Delta u = g$ ← mass density of the moon.

potential

Gravitational Field : $\vec{F} = \nabla u$





Replace the moon with a single point with the same <u>total mass</u>

$$\Delta u_g = M\, \delta(\vec{x} - \vec{x}_0)$$

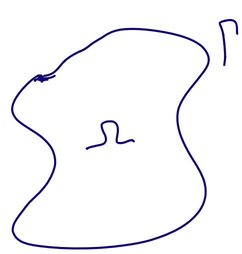
$|u - u_g|$ is <u>very</u> <u>small</u> when evaluated on earth.

Boundary Value Problem:

$\Delta u = 0$ in Ω

$u = f$ on $\Gamma = \partial\Omega$

Interior Dirichlet Problem

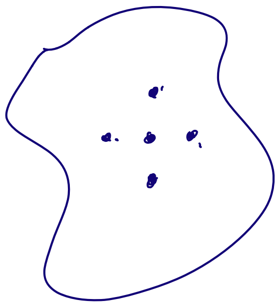


A PDE discretization looks like:

$$\Delta u(\vec{x}) = u_{xx} + u_{yy}$$

$$= \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}$$

5 point stencil



The unknowns in my linear system are the values of u.

The right hand side is mostly zeros.

$$\frac{1}{h^2}\left(\begin{array}{ccccccccc} -4 & 1 & & & & & 1 & & \\ 1 & -4 & 1 & & & & & 1 & \\ & 1 & -4 & 1 & & & & & 1 \\ & & 1 & \ddots & 1 & & & & \\ & & & & \ddots & & & & \\ 1 & & & & 1 & -4 & 1 & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \end{array}\right) \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \\ u_n \end{pmatrix} = \left(\begin{array}{c} \vert \\ 0 \\ \vert \\ \hline f \end{array}\right)$$

$A \approx \Delta$

add in constraints for $u = f$ on the boundary.

The matrix A is sparse. Only $O(1) \approx 5$ entries per column are non-zero.

A can be applied very fast $\Rightarrow$ Good for iterative solvers. $O(n)$

Also, there exist a fast algorithm (nested dissection) for reordering the columns and directly computing A^{-1}. $O(n \log n)$

Integral Equation discretization

A Green's Function for the differential operator $\mathcal{L}$ is the function G such that

$$\mathcal{L} G = \delta$$

A dirac delta function δ is such that

$$f(x) = \int_{\mathbb{R}^2} f(y)\, \delta(x-y)\, dy$$

Example: $\Delta G = \delta$.

$$G = \Delta^{-1} \delta .$$

In 2D for Laplace:

$$\Delta_{\vec{y}} \frac{1}{2\pi} \log \|\vec{x}-\vec{y}\| = \delta(\vec{x}-\vec{y})$$

In 3D: $\Delta_{\vec{y}} \dfrac{1}{4\pi\|\vec{x}-\vec{y}\|} = \delta(\vec{x}-\vec{y})$

One consequence of a Green's Function:

Free space Poisson problem: $\Delta u = g$

The solution can immediately written down by convolving both sides with the Green's Function.

$$\int G(\vec{x}-\vec{y})\, \Delta \vec{u}(\vec{y})\, d\vec{y} = \int G(\vec{x}-\vec{y})\, g(y)\, d\vec{y}$$

By integration by parts

$$\vec{u}(\vec{x}) = \int G(\vec{x}-\vec{y})\, g(\vec{y})\, d\vec{y}.$$

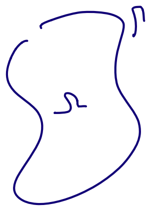
Greens Functions give us ways to represent solutions to PDEs:

Greens Third Indentity: If $\Delta u = 0$ in Ω, then

$$u(x) = \int_\Gamma \frac{\partial G}{\partial \eta_y}(x-y)\, u(y)\, dy - \int_\Gamma G(x-y) \frac{\partial u}{\partial n}(y)\, dy$$

In general, this fact might lead us to represent solutions to PDEs in terms of integrals of the Greens Function..

Poisson BVP:



$$\Delta u = g \quad \text{in } \Omega$$

$$u = f \quad \text{on } \Gamma$$

Represent $\quad (*) \quad u = \underbrace{\int_\Omega G(x-y)\, g(y)\, dy}_{\text{automatically satisfies the PDE, but not the boundary condition}} + \underbrace{\int_\Gamma \frac{\partial G}{\partial n_y}(x-y)\, \overset{\text{unknown}}{\sigma(y)}\, dy}_{\text{Double layer potential } \mathcal{D}\sigma}$

To derive an integral equation, insert $(*)$ into the boundary condition.

$$\Rightarrow \frac{1}{2}\sigma(x) + \int_\Gamma \frac{\partial G}{\partial n_y}(x-y)\, \sigma(y)\, dy = f - \int_\Omega G(x-y)\, g(y)\, dy.$$

When discretized, we get

$$\left(\frac{1}{2} I + K\right)\sigma = RHS$$

K ← this matrix is <u>dense</u>.

Apply: $O(n^2)$.

Invert: $O(n^3)$. where n is the number of points on the boundary.

Advantages of IEs:

- Mathematically better: ① well-conditioned (small condition number)
  ② Excellent for exterior problems

Disadvantages ① Linear systems are dense

② Requires singular integrals.