Fast Multipole Methods

At their core, FMMs are used for computing

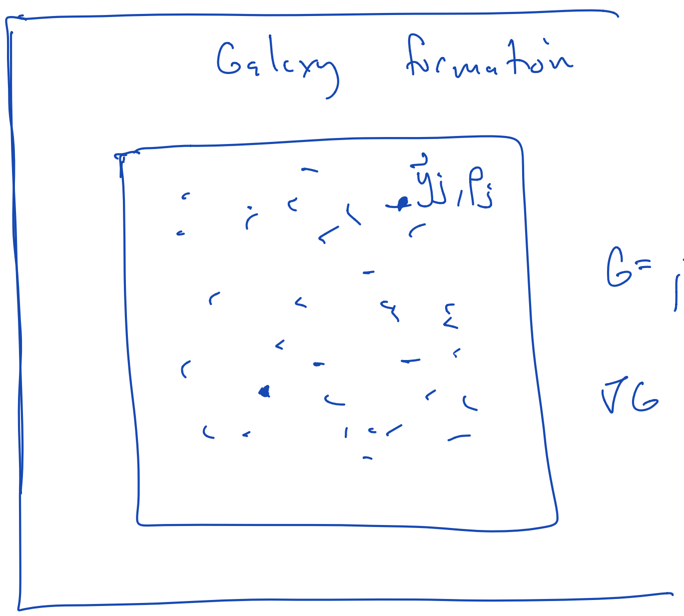
$$u_i = \sum_{j=1}^{N} G(\vec{x}_i, \vec{y}_j)\, q_j \qquad \text{N-body problem}$$

$\vec{x}_i$'s are "targets"

$\vec{y}_j$'s are "sources"

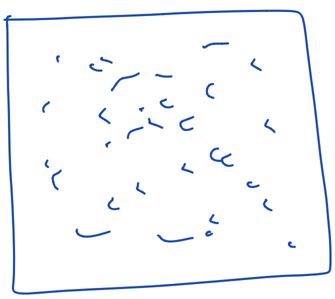
G kernel of interaction

u_i potential at $\vec{x}_i$



$G = \dfrac{1}{|\vec{x}-\vec{y}|}$

∇G



N points, $\vec{x}_i$

$$u_i = \sum_{j=1}^{N} G(\vec{x}_i, \vec{x}_j)\, q_j$$

Cost: $\mathcal{O}(N^2)$

Consider $\vec{x}_i \in \mathbb{R}^2$, $G(\vec{x},\vec{y}) = \log|\vec{x}-\vec{y}|$.

$$\Delta G^{(x-y)} = 2\pi\, \delta(x-y).$$

Main idea: Separate variables

Sources Targets

Goal: Write $G(\vec{x}, \vec{x}') \approx \sum_{l=0}^{p} B_l(\vec{x})\, C_l(\vec{x}')$

If this is possible, then our sum becomes:

$$u_i = \sum_j G(\vec{x}_i, \vec{x}_j)\, q_j$$

$$\approx \sum_j \left(\sum_{l=0}^{p} B_l(\vec{x}_i)\, C_l(\vec{x}_j) \right) q_j$$

$$= \sum_{l=0}^{p} B_l(\vec{x}_i) \underbrace{\sum_{j=1}^{N} C_l(\vec{x}_j)\, q_j}$$

For each l, compute sum: $O(Np)$

Call them M_l.

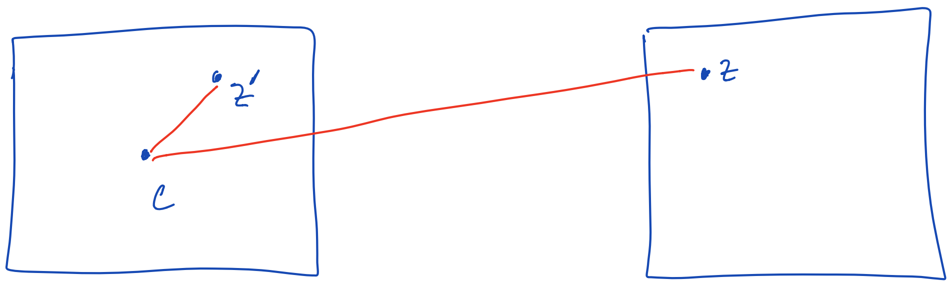
$$u_i = \underbrace{\sum_{l=0}^{p} M_l B_l(\vec{x}_i)}.$$

Evaluating at every $\vec{x}_i$ cost $O(Np)$.

Cost: $O(N^2) \to O(2Np)$.

Consider the complex kernel $\log(z)$. when $z = x + iy$

then $\mathrm{Re}(\log z) = \log|z|$



Sources

Targets

In what follows $|z' - c| < \beta |z - c|$ with $\beta < 1$

Also, recall that

$$\log(1-z) = -\sum_{l=1}^{\infty} \frac{z^l}{l} \quad \text{for } |z| < 1$$

This means that

$$\log(z - z') = \log\left(z - c - (z'-c)\right)$$

$$= \log\left(z - c\left(1 - \frac{z'-c}{z-c}\right)\right)$$

$$= \log(z-c) + \log\left(1 - \frac{z'-c}{z-c}\right)$$

$$= \log(z-c) - \sum_{l=1}^{\infty} \frac{1}{l}\left(\frac{z'-c}{z-c}\right)^l .$$

In polar coordinates

$z' - c = r' e^{i\theta'}$

$z - c = r e^{i\theta}$

$$= \log(z-c) - \sum_l \frac{1}{l}\left(\frac{r'}{r}\right)^l e^{il(\theta'-\theta)}$$

$$= \log(z-c) - \sum_l \frac{1}{l} \boxed{\frac{e^{-il\theta}}{r^l}} \boxed{r'^l e^{il\theta'}}$$

Approximate log using this expression: Truncate after p terms:

$$\left| \log(z-z') - \left(\log(z-c) - \sum_{l=1}^{p} \frac{1}{l} \frac{e^{-il\theta}}{r^l} r'^l e^{il\theta'} \right) \right|$$

$$= \left| \sum_{l=p+1}^{\infty} \frac{1}{l} \left(\frac{z'-c}{z-c} \right)^l \right|$$

$$\leq \frac{1}{p+1} \sum_{l=p+1}^{\infty} \left| \frac{z'-c}{z-c} \right|^l \leq \frac{1}{p+1} \sum_{l=p+1}^{\infty} \beta^l = \frac{1}{p+1} \frac{\beta^{p+1}}{1-\beta}$$

Insert a truncated approximation into our sum:

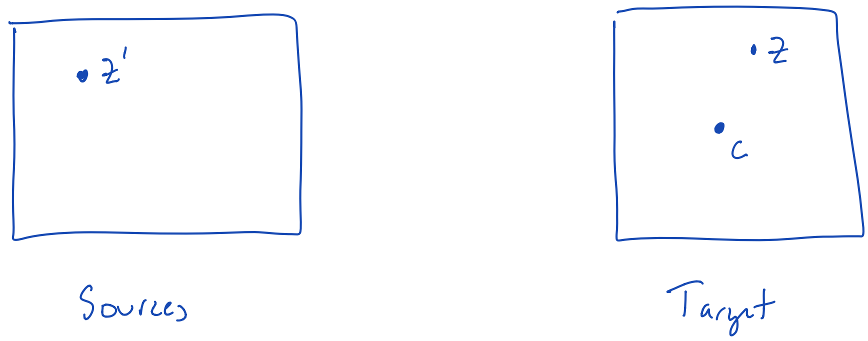
$$u_i \approx \sum_{j=1}^{N} q_j \left(\log(z_i-c) - \sum_{l=1}^{p} \frac{1}{l} \frac{e^{-il\theta_i}}{r_i^l} r'^l e^{il\theta'} \right)$$

$$= \log(z_i-c) \underbrace{\sum_{j=1}^{N} q_j}_{M_0} - \sum_{l=1}^{p} \frac{1}{l} \frac{e^{-il\theta_i}}{r_i^l} \underbrace{\left(\sum_{j=1}^{N} q_j r'^l e^{il\theta'} \right)}_{M_l}$$

$$= M_0 \log(z_i-c) - \sum_{l=1}^{p} \frac{M_l}{l} \frac{e^{-il\theta_i}}{r_i^l}$$

These coefficients M_l are known as "multipole" coefficients or "outgoing" coefficients.

Likewise, we can expand in the target box:



$|z-c| < \beta |z'-c|$

$$\log(z-z') = \log\big(z-c-(z'-c)\big)$$

$$= \log\left(z'-c \left(\frac{z-c}{z'-c} - 1 \right) \right)$$

$$= \log(z'-c) + \underbrace{\log\left(\frac{z-c}{z'-c} - 1 \right)}$$

$$= \log(z'-c) + \sum_{l=1}^{\infty} \frac{1}{l} \left(\frac{z-c}{z'-c} \right)^l$$

$$\simeq \log(z'-c) + \sum_{l=1}^{p} \frac{1}{l} \left(\frac{z-c}{z'-c} \right)^l + \mathcal{O}\left(\beta^{p+1}\right).$$

$$u_i \approx \sum_{j=1}^{N} q_j \left(\log(z_j-c) + \sum_{l=1}^{p} \frac{1}{l} \left(\frac{z_i-c}{z_j-c} \right)^l \right)$$

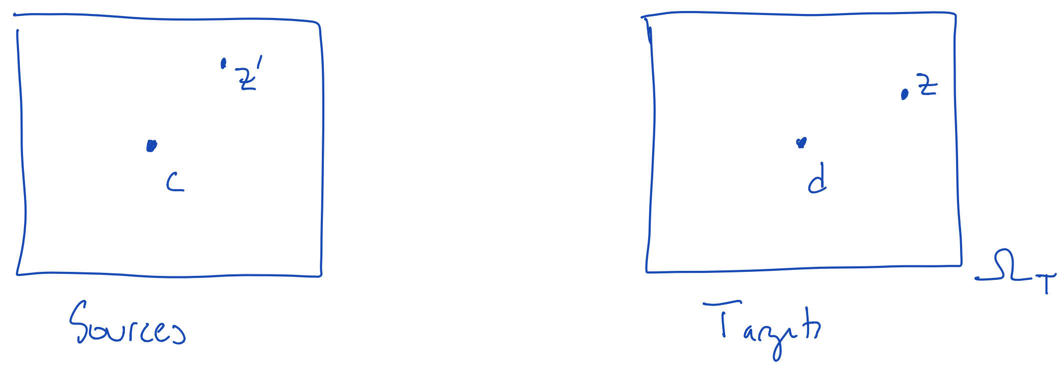
$$= \underbrace{\sum_{j=1}^{N} q_j \log(z_j-c)}_{L_0} + \sum_{l=1}^{p} \frac{1}{l} (z_i-c)^l \underbrace{\sum_{j=1}^{N} q_j \frac{1}{(z_j-c)^l}}_{L_l}$$

$$= L_0 + \sum_{l=1}^{p} \frac{L_l}{l} (z_i-c)^l$$

"Local" expansion

"Incoming" expansion.

Multipole to Local Translation.



The potential in Ω_T can be written in one of two ways:

$$u(z) \approx \sum_{\ell=0}^{p} \frac{M_\ell}{\ell} \frac{1}{(z-c)^\ell}$$

$$\approx \sum_{\ell=0}^{p} \frac{L_\ell}{\ell} (z-d)^\ell$$

Idea is to write $\frac{1}{(z-c)^\ell}$ in terms of $(z-d)^\ell$

Fact

$$\frac{1}{(z-c)^\ell} = \frac{1}{\left((d-c)-(d-z)\right)^\ell} = \frac{1}{(d-c)^\ell} \frac{1}{\left(1-\frac{d-z}{d-c}\right)^\ell}$$

$$= \pm \frac{1}{(d-c)^\ell} \sum_{r=0}^{\infty} \binom{\ell+r-1}{r-1} \left(\frac{z-d}{c-d}\right)^r$$

Translation operation "Multipole to local"