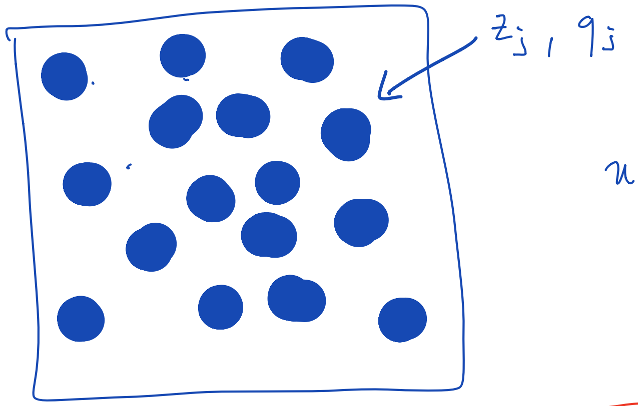
Fast Solvers

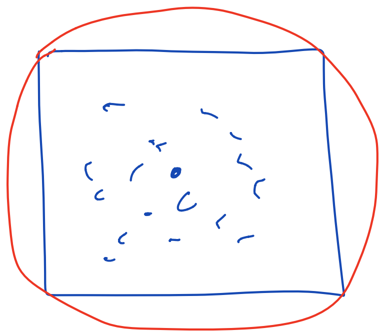
Oct 7, 2020

Computational task:



$$u(z_i) = u_i = \sum_{j=1}^{N} q_j \log(z_i - z_j).$$

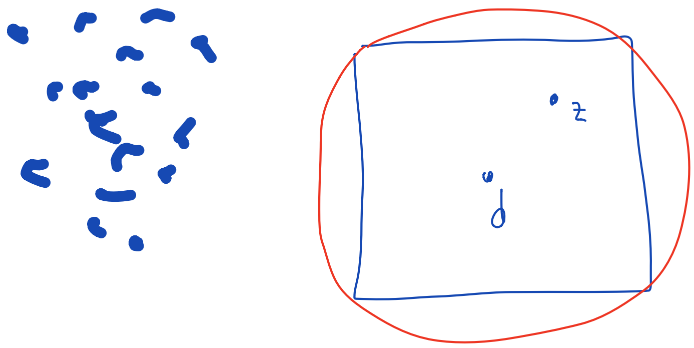
Multipole expansions:



Write $u(z)$ outside this disk as

$$u(z) = M_0 \log(z-c) + \sum_{\ell=1}^{\infty} M_\ell \frac{1}{(z-c)^\ell}.$$

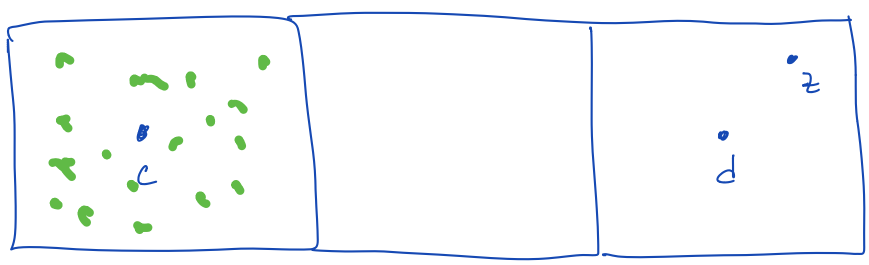
Local Expansions



Inside this disk we can write

$$u(z) = \sum_{\ell=0}^{\infty} L_\ell (z-d)^\ell$$

For finite precision calculations, truncate each expansion after p terms.



$$u(z) \approx \boxed{M_0} \log(z-c) + \sum_{\ell=1}^{p} \boxed{M_\ell} \frac{1}{(z-c)^\ell}$$

p coefficients

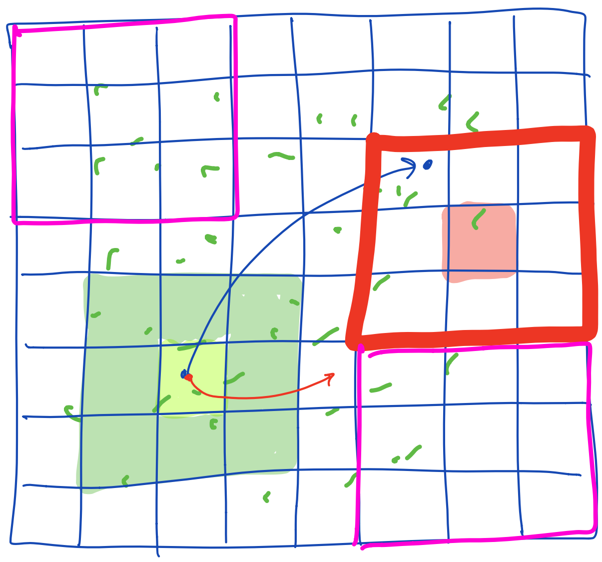
$$\approx \sum_{\ell=0}^{p} \boxed{L_\ell} (z-d)^\ell .$$

There is a linear operator T_{M2L} which maps the M_ℓ's to the L_ℓ's:

$$\begin{pmatrix} L_0 \\ \vdots \\ L_p \end{pmatrix} = T_{M2L} \begin{pmatrix} M_0 \\ \vdots \\ M_\ell \end{pmatrix}$$

The entries of T_{M2L} are known.

A single level scheme



(0) Divide the region into m equal-sized boxes, N/m points per box.
(assume uniform dist. points)

(1) Form multipole expansions for each box, $O(Np)$ flops.

(2) Translate all multipole expansions to local expansions, ignoring neighbor boxes. $O(m^2 p^2)$ flops.

(3) Evaluate the potential

$$O\left(N \cdot \frac{N}{m}\right) + O(Np) = O\left(N^2/m + Np\right)$$

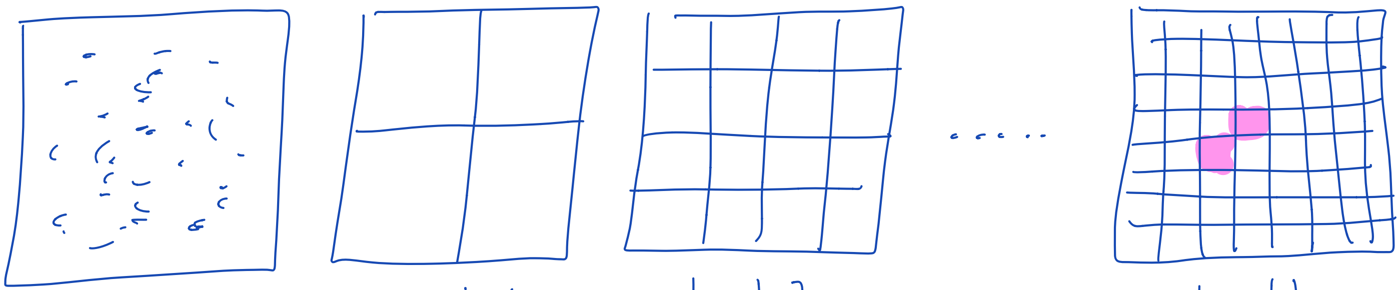
Total cost: $O\left(2Np + N^2/m + m^2 p^2\right)$

Choose m to minimize this cost

$$-\frac{N^2}{m^2} + 2mp^2 = 0 \quad \Rightarrow \quad m \sim N^{2/3}$$

$$\Rightarrow \quad \mathcal{O}\left(2Np + N^{4/3} + N^{4/3}\right) \quad \sim \quad \mathcal{O}\left(N^{4/3}\right)$$

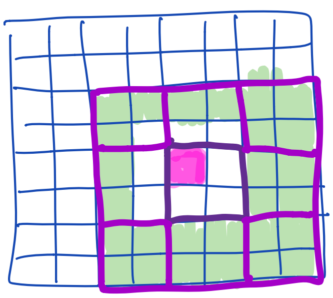
Fast Multipole Method



Level 0   Level 1   Level 2   . . . . .   Level L

$L \sim \log_4 N$.

- Parent and children boxes.
- Neighbor box : Two boxes are neighbors if they are on the same level and share an edge or corner.
- Interaction list :



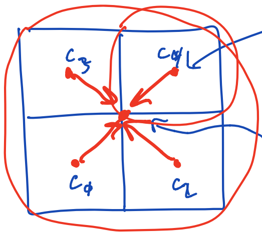
(green) Is the interaction list for (pink)

Note the multipole expansion for (pink) is accurate everywhere in the interaction list.

(0) Form dyadic tree structure with 4^L boxes on finest level L.

(1) P2M on level L $\mathcal{O}(Np)$ $(L \sim \log_4 N)$

(2) Upward pass : From level L to 3, translate multipole expansion from box to its parent.



c = center of parent

$$u \sim M_0^4 \log(z-c_4) + \sum_{\ell=1}^{p} M_\ell^4 \frac{1}{(z-c_4)^\ell}$$

$$u \sim M_0 \log(z-c) + \sum_{\ell=1}^{p} M_\ell \frac{1}{(z-c)^\ell}$$

4 boxes on level L

$$\begin{pmatrix} M_0 \\ M_1 \\ \vdots \\ M_p \end{pmatrix} = T_{M2M} \begin{pmatrix} M_0^4 \\ M_1^4 \\ \vdots \\ M_p^4 \end{pmatrix}$$

Cost: On level ℓ, there are 4^ℓ boxes, and maximum depth is L.

Cost of M2M on level ℓ is $p^2 4^\ell$.

$$\Rightarrow \sum_{\ell=3}^{L} p^2 4^\ell \sim$$

If $O(1)$ points per box on level L, and level L has 4^L boxes, then $4^L \sim N$.

$$\sim O(p^2 4^L) \sim O(p^2 N).$$

$O(p^2 N)$

Sideways pass

(3) For each box on levels $2, \ldots, L$, translate its multipole expansion to a local expansion in each box of its interaction list. The interaction list has 27 boxes.

Cost: $\boxed{O(27 p^2 N)}$.

(4) Downward pass

From level 2 to level L-1, translate each box's local expansion to a local expansion of its child.

Cost: $O(p^2 N)$

5. Evaluate the potential.

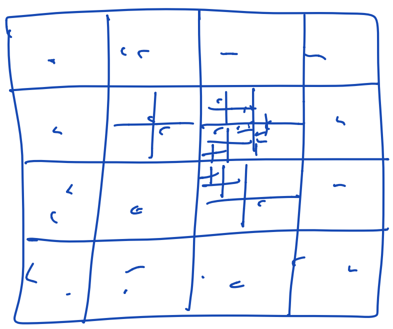
(a) Near-field direct calculation $\mathcal{O}(N)$

(b) Evaluating the local expansions $\mathcal{O}(pN)$

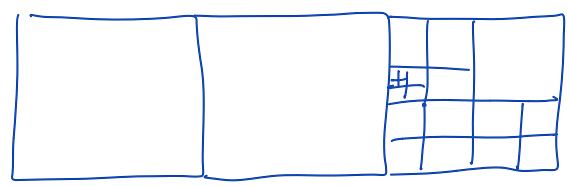
Overall cost is $\mathcal{O}(N)$.

Extensions

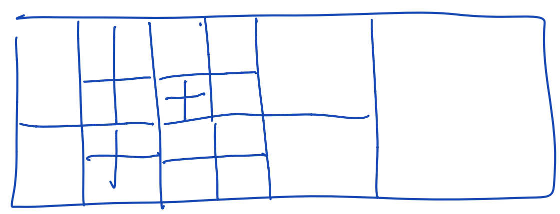
1. Adaptive trees.



Unrestricted adaptive tree



Level-restricted tree



2. Other PDEs.

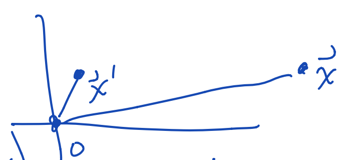
All an FMM needs are "outgoing" and "incoming" expansions.

Laplace in 3D:

$$\Delta u = 0 \quad \text{in } 3D.$$

$$\sum_{j \neq i} \frac{q_j}{4\pi |\vec{x}_i - \vec{x}_j|} = u_i$$

$$G = \frac{1}{4\pi r} = \frac{1}{4\pi |\vec{x} - \vec{x}'|}$$



$$\frac{1}{4\pi |\vec{x} - \vec{x}'|} \approx \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r'^{l}}{r^{l+1}} Y_l^m(\theta, \varphi) Y_l^{-m}(\theta', \varphi')$$

$\vec{x} = (r, \theta, \varphi)$

$\vec{x}' = (r', \theta', \varphi')$

$$Y_\ell^m(\theta,\varphi) = \text{spherical harmonic}$$
$$= P_\ell^m(\cos\theta)\, e^{im\varphi}.$$

Outgoing representations : $\sim \sum_{l=0}^{p} \sum_{m=-l}^{l} \alpha_{lm} \frac{1}{r^{l+1}} Y_l^m(\theta,\varphi)$

Incoming representations : $\sim \sum_{l=0}^{p} \sum_{m=-l}^{l} \beta_{lm}\, r^l\, Y_l^m(\theta,\varphi)$.

Helmholtz:

$(\Delta + k^2)u = 0$ in 3D.

Green's Function : $G_k(\vec{x},\vec{y}) = \dfrac{e^{ik|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|}$

$$\frac{e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|} \approx \sum_{l=0}^{\infty} \sum_{m=-l}^{l} j_l(kr')\, h_l(kr)\, Y_l^m(\theta,\varphi)\, Y_l^{-m}(\theta',\varphi')$$

spherical Bessel functions

$j_\ell = J_{\ell+1/2} \quad h_\ell = H_{\ell+1/2}$.