Theory of Probability Sep 9, 2020

Counting

① Roll a die : Outcomes are 1, 2, 3, 4, 5, 6

② Roll a die, choose a card from playing deck : Die : 6 possibilities

Cards : 52 distinct cards.

Total number of die - card pairs :

$6 \cdot 52 = 312$ possibilities

In general, if we have M experiments, each with n_i distinct outcomes, then we have

$n_1 \cdot n_2 \cdot n_3 \cdot n_4 \cdots n_M$ total possible outcomes.

Back to our example, $n_1 = 6$

$n_2 = 52.$

Permutations

Ordering distinct objects.

Example: How many ways can I order 6 numbers:

Fill in the blanks : _ _ _ _ _ _

$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ ← possibilities

$= 6! = 720$

Factorial : $n! = n\cdot(n-1)\cdot(n-2)\cdots\underbrace{(n-(n-1))}_{1}$

$$= \prod_{j=1}^{n} j$$

Convention : $0! = 1$

Example 11 people on a soccer team, each person plays one position.

$11\cdot 10 \cdots 2\cdot 1 = 11! = 39{,}916{,}800$

Example Given 4 textbooks on math, 3 on English.

7 total textbooks $\Rightarrow$ 7! possible orderings of all 7 textbooks

Sub-example If all math textbooks come first, then English textbooks:

$$\underbrace{\underline{M}\ \underline{M}\ \underline{M}\ \underline{M}}_{4! = 24}\ \underbrace{\underline{E}\ \underline{E}\ \underline{E}}_{3! = 6} \Rightarrow 4!\,3! = 24\cdot 6 = 144 \text{ orderings}$$

Example How many orderings of the letters in PEPPER are there?

If each letter is assumed to be distinct, then are $6! = 720$ possibilities.

If the E's are not distinct, then we must divide 720 by the number of permutations of these E's:

$$\frac{720}{2!} = 360 = \frac{6!}{2!}$$

Likewise, if the P's are not distinct, then we must divide by the number of permutations of this letter $\rightarrow$ 3!.

So the number of ~~distinct~~ words arrangements that can be made using the letters in PEPPER is:

$$\frac{6!}{2!\,3!} = \frac{\overset{3}{\cancel{6}} \cdot 5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot 1 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 60.$$

In general, if we have n objects, of which n_1 are alike, n_2 are alike, ... n_r are alike, then the number of distinct permutations is

$$\frac{n!}{n_1!\, n_2! \cdots n_r!}.$$

Combinations: How many distinct unordered groups can be formed from a set of objects.

Example: Take the letters A, B, C, D, E.

How many groups of 3 letters can be form?

$$\frac{5 \cdot 4 \cdot 3}{3!} = \frac{60}{6} = 10.$$

In general, the number of unordered sets of r distinct objects that can be formed from n distinct elements is

$$\binom{n}{r} = \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-r+1)}{r!} \quad \frac{(n-r)(n-r-1)\cdots 1}{\underbrace{(n-r)\cdots 1}_{(n-r)!}}$$

$$= \frac{n!}{(n-r)!}\,\frac{1}{r!}$$

$$= \frac{n!}{(n-r)!\,r!} \qquad \text{"}n \text{ choose } r\text{".}$$

Application: <u>Binomial Theorem</u>

<u>Binomial Thm</u>: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ $(*)$

Ex: $(x+y)^2 = x^2 + 2xy + y^2$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

<u>Proof</u> (By induction)

① Prove for $n = 0, 1, 2$ proved by calculation

② Assume $(*)$ is true for $n = m$.

③ Show that $(*)$ also holds for $n = m+1$.

Show is that

$$(x+y)^{m+1} = (x+y)\underbrace{(x+y)^{m}} = \sum_{k=0}^{m+1}\binom{m+1}{k} x^{k} y^{m+1-k}$$

= (by assumption)

$$= (x+y)\sum_{k=0}^{m}\binom{m}{k} x^{k} y^{m-k}$$

$$= \sum_{k=0}^{m}\binom{m}{k} x^{k+1} y^{m-k} + \sum_{k=0}^{m}\binom{m}{k} x^{k} y^{m+1-k}$$

$$= \sum_{k=1}^{m+1}\binom{m}{k-1} x^{k} \underbrace{y^{m-(k-1)}}_{y^{m+1-k}} + \sum_{k=0}^{m}\binom{m}{k} x^{k} y^{m+1-k}$$

$$= x^{m+1} + \sum_{k=1}^{m}\binom{m}{k-1} x^{k} y^{m+1-k} + y^{m+1} + \sum_{k=1}^{m}\binom{m}{k} x^{k} y^{m+1-k}$$

$$= x^{m+1} + \sum_{k=1}^{m}\underbrace{\left(\binom{m}{k-1}+\binom{m}{k}\right)} x^{k} y^{m+1-k} + y^{m+1}$$

$$\underbrace{\frac{m!}{(m-(k-1))!}\cdot\frac{1}{(k-1)!} + \frac{m!}{(m-k)!\,k!}}$$

$$\frac{m!}{(m+1-k)!\,(k-1)!} + \frac{m!}{(m-k)!\,k!} = \frac{m!\cdot k}{(m+1-k)!\,k!} + \frac{m!\,(m+1-k)}{(m+1-k)!\,k!}$$

$$= \frac{\cancel{m!\,k} + m!\,(m+1) - \cancel{m!\,k}}{(m+1-k)!\;k!}$$

$$= \frac{(m+1)!}{(m+1-k)!\;k!} = \binom{m+1}{k}$$

In the end we have:

$$\Rightarrow \underbrace{x^{m+1}}_{\text{corresponds to } \binom{m+1}{m+1} x^{m+1} y^{0}} + \sum_{k=1}^{m} \binom{m+1}{k} x^{k} y^{m+1-k} + \underbrace{y^{m+1}}_{\text{corresponds to } \binom{m+1}{0} x^{0} y^{m+1-0}}$$

$$= \sum_{k=0}^{m+1} \binom{m+1}{k} x^{k} y^{m+1-k}. \quad \square$$