Theory of Probability

Sep 23, 2020

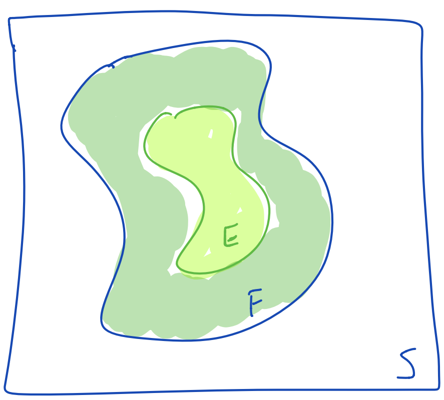
$$P(\text{die comes up} = 3) = 1/6$$

Now imagine that you know that the die comes up odd:

$$P(\text{die} = 3 \mid \text{die came up odd}) = 1/3.$$

Definition "The conditional probability of E given F" (assuming $P(F) > 0$) is:

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$



Example: (For this Venn diagram)

$P(F) = .5$

$P(E) = .25$ | $P(E \cap F) \overset{?}{=} P(E) = .25$

since $E \subset F$.

$$P(E|F) = \frac{.25}{.5} = .5$$

Slightly surprising: $P(E|F) = \dfrac{P(E \cap F)}{P(F)}$

Likewise $P(F|E) = \dfrac{P(E \cap F)}{P(E)}$

Rearrange this

$$P(E \cap F) = P(E|F)\,P(F)$$
$$= P(F|E)\,P(E)$$

these are equal to each other.

Set them equal to each other:

$$P(E|F)\,P(F) = P(F|E)\,P(E)$$

$$\Rightarrow \boxed{P(E|F) = \frac{P(F|E)\,P(E)}{P(F)}}$$

$P(E \cap F)$

Bayes Rule.

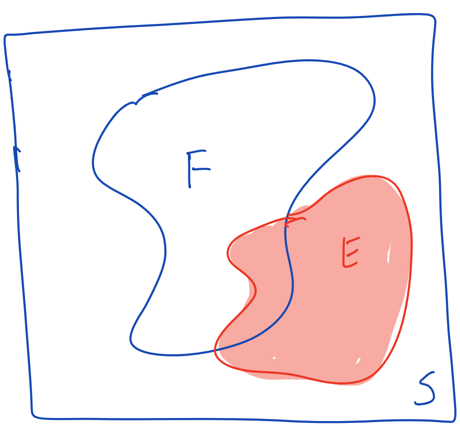
Multiplication Rule :

$$P(EF) = P(F|E)\,P(E)$$
$$= P(E|F)\,P(F)$$

More generally :

$$P(E_1 E_2 E_3 \cdots E_n) = P(E_1)\,P(E_2|E_1)\,P(E_3|E_1E_2) \cdots P(E_n|E_1 \cdots E_{n-1})$$

Furthermore, we have the Law of Total Probability :



The sample space can be split into two mutually exclusive events : $S = E \cup E^c$

$E \cap E^c = \phi$

$$\Rightarrow F = \underline{(E \cap F)} \cup \underline{(E^c \cap F)}$$

Mutually exclusive.

$$\Rightarrow \boxed{P(F) = P(EF) + P(E^cF)}$$

More generally, if $E_1, \ldots, E_n$ are mutually exclusive events and $E_1 \cup E_2 \cup \ldots \cup E_n = S$, then for any event F,

$$F = \bigcup_{i=1}^{n} (F \cap E_i)$$

$$\Rightarrow P(F) = \sum_{i=1}^{n} P(FE_i).$$

Often very useful in Bayes Rule:

$$P(E|F) = \frac{P(F|E)\,P(E)}{P(F)}$$

$$= \frac{P(F|E)\,P(E)}{P(EF) + P(E^cF)}$$

$\leftarrow$ expand these terms using conditional probability

$$= \frac{P(F|E)\,P(E)}{P(E|F)P(F) + P(E^c|F)P(F)}$$

Bayes Rule and the law of total probability.

since $P(EF) = P(E|F)P(F)$
$= P(F|E)P(E)$

$$= \frac{P(F|E)\,P(E)}{P(F|E)P(E) + P(F|E^c)P(E^c)}$$

<u>Example</u> Let E = event you are actually COVID positive

F = event you <u>test</u> positive.

$$P(E \mid F) = \frac{P(F \mid E)\, P(E)}{P(F)}$$

$$= \frac{P(F \mid E)\, P(E)}{P(FE) + P(FE^c)}$$

$$= \frac{P(F \mid E)\, P(E)}{P(F \mid E)\, P(E) + P(F \mid E^c)\, P(E^c)}$$

Lets consider these probabilities:

E (actually positive)

F (test positive)		E	E^c
	F	.99	.05
	F^c	.01 <u>False negative</u>	.95

$P(E) = .0001$

$$= \frac{.99 \times .0001}{.99 \times .0001 + .05 \times .9999}$$

$$= \frac{.000099}{.000099 + .049995} = \frac{.000099}{.050094} = \approx 002$$