Theory of Probability

Sep 30, 2020

Two events $E, F \subset S$, then the conditional probability of E given F is

$$P(E|F) = \frac{P(EF)}{P(F)}$$

Fix the event F, and consider the function

$$Q(E) = P(E|F).$$

It turns out that $Q(\cdot) = P(\cdot | F)$ satisfies the three axiom of probability:

① $0 \leq Q(E) \leq 1$

Pf: $0 \leq P(E|F) \leq 1$

$$0 \leq \frac{P(EF)}{P(F)} \leq 1$$

$$\Rightarrow 0 \leq P(EF) \leq P(F)$$

satisfied because $EF \subseteq F$

② $P(S|F) = 1 = Q(S)$

$$P(S|F) = \frac{P(SF)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

③ Let E_i be mutually exclusive events: The events E_iF are mutually exclus.

$$Q\left(\bigcup_i E_i\right) = \sum_i Q(E_i)$$

$$P(\cup E_i | F) = \frac{P(\cup E_i \cap F)}{P(F)} = \frac{P(\bigcup_i E_iF)}{P(F)} = \frac{\sum P(E_iF)}{P(F)}$$

$$= Q(E_i)$$

$Q(E) = P(E|F)$ define a new probability function.

Consider conditional probabilities under Q:

$$Q(E_1|E_2) = \frac{Q(E_1E_2)}{Q(E_2)} = \frac{P(E_1E_2|F)}{P(E_2|F)}$$

$$= \frac{P(E_1E_2F)}{P(F)} \quad \frac{P(F)}{P(E_2F)} = \frac{P(E_1E_2F)}{P(E_2F)}$$

Conditional Independence

Definition: E_1 and E_2 are conditionally independent with respect to F if:

$$P(E_1|E_2F) = P(E_1|F).$$

Very similar.

Independence means: $P(A|B) = P(A)$.

An equivalent definition of conditional independence is:

$$P(E_1E_2|F) = P(E_1|F)\,P(E_2|F)$$

(compare with $P(AB) = P(A)\,P(B)$ if A, B are independent).

Pf of equivalence:

$$P(E_1|E_2F) = \frac{P(E_1E_2F)}{P(E_2F)}$$

$$= \frac{P(E_1E_2|F)\,P(F)}{P(E_2F)}$$

Since $P(E_1 | E_2 F) = P(E_1 | F)$, we have that

$$P(E_1 | F) = \frac{P(E_1 E_2 | F)\, P(F)}{P(E_2 F)}$$

$$P(E_1 E_2 | F) = P(E_1 | F) \underbrace{\frac{P(E_2 F)}{P(F)}}_{= P(E_2 | F)}$$

$$\underline{P(E_1 E_2 | F) = P(E_1 | F)\, P(E_2 | F)}$$