Theory of Probability

Oct 21, 2020

① $X \sim \text{Geometric}(p)$ if

$$P[X = k] = (1-p)^{k-1} p \qquad \text{for} \quad k = 1, 2, \ldots$$

"$k-1$ failures, before a success on the k^{th} trial"

$$E[X] = \frac{1}{p} \qquad Var[X] = \frac{1-p}{p^2}.$$

② $X \sim \text{Negative Binomial}(r, p)$

$$\underbrace{\underline{T}\ \underline{T}\ \underline{T}\ \underline{H}\ \underline{T}\ \underline{T} \ldots \underline{H}\ \underline{T}\ \underline{T} \ldots \underline{H}}_{n \text{ flips}}$$

r heads

$n-r$ tails.

$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

$$E[X] = \frac{r}{p} \qquad Var[X] = \frac{r(1-p)}{p^2}.$$

③ $X \sim \text{Hypergeometric}(N, m, n)$

Imagine an urn with N balls, m are white, $N-m$ are blue, randomly choose n of them.

$$P[W = k] = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$$

$$E[W] = \frac{mn}{N}, \quad Var[W] = \frac{mn}{N}\left(\frac{(m-1)(n-1)}{N-1} + \frac{N-mn}{N}\right)$$

Sums of Random Variables (§4.9)

Consider two random variables X, Y.

then $Z = X + Y$ is also a random variable.

Let $s \in S$ be an individual outcome in the sample space,

$X(s)$ = the value that X takes when s occurs.

$Y(s) = \ldots$

$\Rightarrow \quad Z(s) = X(s) + Y(s).$

$$E[X] = \sum_i x_i \, P[X = x_i]$$

For each i, let

S_i = set of outcomes for which $X = x_i$

$= \{ s \in S : X(s) = x_i \}.$

$$= \sum_i x_i \, P[S_i]$$

$$= \sum_i x_i \sum_{s \in S_i} p(s)$$

$p(s) = P[s]$

$$= \sum_i \sum_{s \in S_i} x_i \, p(s)$$

$$= \sum_i \sum_{s \in S_i} X(s) \, p(s)$$

$$\boxed{= \sum_{s \in S} X(s) \, p(s)}$$

since one could show that S_i are mutually exclusive.

Next consider computing $E[X+Y] = E[Z]$

By the previous calculation,

$$E[Z] = \sum_{s\in S} Z(s)\, p(s).$$

$$= \sum_{s\in S} \big(X(s) + Y(s)\big)\, p(s)$$

$$= \sum_{s\in S} X(s)\,p(s) + \sum_{s\in S} Y(s)\,p(s)$$

$$= E[X] + E[Y].$$

Cumulative Distribution Function

$F(x) = P[X \le x]$.

① F is non-decreasing $\Rightarrow$ if $a \le b$, then $F(a) \le F(b)$.

② $\lim_{b\to\infty} F(b) = \lim_{b\to\infty} P[X \le b] = 1$

③ $\lim_{a\to -\infty} F(a) = 0$

④ F is <u>right</u> continuous.

If $b_n \to b$, with $b_{n+1} \le b_n$, then $\lim_{n\to\infty} F(b_n) = F(b)$.

$P[X=0] = \frac{1}{2}$

$P[X=1] = \frac{1}{2}$