Theory of Probability Oct 26, 2020

X is a continuous random variable if there exists a nonnegative f, defined for all $x \in (-\infty, \infty]$ such that

$$P[X \in B] = \int_B f(x)\, dx.$$

with B any set of real numbers.

$$P[X \in [a,b]] = \int_a^b f(x)\, dx$$

$$P[\underset{[a,a]}{\underbrace{X = a}}] = \int_a^a f(x)\, dx = 0.$$

Cumulative Dist Function

$$P[X \leq x] = \int_{-\infty}^{x} f(t)\, dt. = F(x).$$

Ex: $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \quad \lambda > 0 \\ 0 & x < 0 \end{cases}$

$$\int_0^\infty f(x)\, dx = \int_0^\infty \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_0^\infty$$

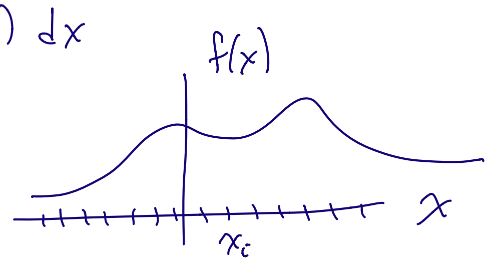
$$= 0 - (-1) = 1$$

By the Fundamental Theorem of Calculus:

$$\frac{d}{dx} P[X \leq x] = \frac{d}{dx} F(x) = \frac{d}{dx} \int_{-\infty}^{x} f(t)\, dt = f(x). \Rightarrow F' = f.$$

Expectation

Since $P[x \le X \le x + dx] \approx f(x)\,dx$



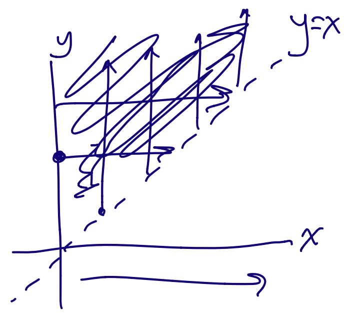
$$\Rightarrow \quad E[X] = \sum_i x_i f(x_i)\,dx$$

$$= \int_{-\infty}^{\infty} x\, f(x)\,dx$$

Lemma: If $X \geq 0$, then $E[X] = \int_0^{\infty} P[X > x]\,dx$.

Proof

$$\int_0^{\infty} P[X \geq x]\,dx = \int_0^{\infty} \underbrace{\int_x^{\infty} f(y)\,dy}\,dx$$



$$= \int_0^{\infty} \int_0^{y} f(y)\,dx\,dy$$

$$= \int_0^{\infty} f(y) \int_0^{y} dx\,dy$$

$$= \int_0^{\infty} f(y)\, y\,dy = E[X].$$

Proposition $E[g(X)] = \int g(x) f(x)\,dx.$

Just as before (for discrete random variables)

$$E[aX + b] = \int (ax + b) f(x)\,dx$$

$$= a \int x f(x)\,dx + b \underbrace{\int f(x)\,dx}_{1}.$$

$$= aE[X] + b.$$

And the variance of X is defined analogously:

$$\mathrm{Var}[X] = E[(X-\mu)^2] \quad \text{where } \mu = E[X]$$

$$= \int (x-\mu)^2 f(x)\,dx.$$

$$= E[X^2] - \underbrace{(E[X])^2}_{\mu^2}.$$