<u>Theory of Probability</u> <u>Oct 28, 2020</u>

The uniform random variable is one that takes on values in an interval with equal probability:

$$\mathcal{U} \sim \text{Uniform}(0,1)$$

$$f(x) = \begin{cases} 1 & \text{on } (0,1) \\ 0 & \text{otherwise} \end{cases}$$

If $0 < a < b < 1$ then

$$P[\mathcal{U} \in (a,b)] = \int_a^b f(x)\,dx$$

$$= \int_a^b dx = b-a$$

$$\Rightarrow F(x) = P[\mathcal{U} \leq x]$$

$$= \begin{cases} 0 & x \leq 0 \\ x & x \in (0,1) \\ 1 & x \geq 1 \end{cases}$$

The uniform density can be defined on any interval:

$$f(u) = \begin{cases} \dfrac{1}{\beta-\alpha} & \text{for } u \in (\alpha,\beta) \\ 0 & \text{otherwise} \end{cases}$$

$$\underset{\alpha < a < b < \beta}{P[\mathcal{U} \in (a,b)]} = \int_a^b \frac{1}{\beta-\alpha}\,dx = \frac{b-a}{\beta-\alpha}$$

Expectation of Uniform:

$$E[U] = \int_0^1 u \, du$$

$$= \frac{1}{2}u^2 \Big|_0^1 = \frac{1}{2}$$

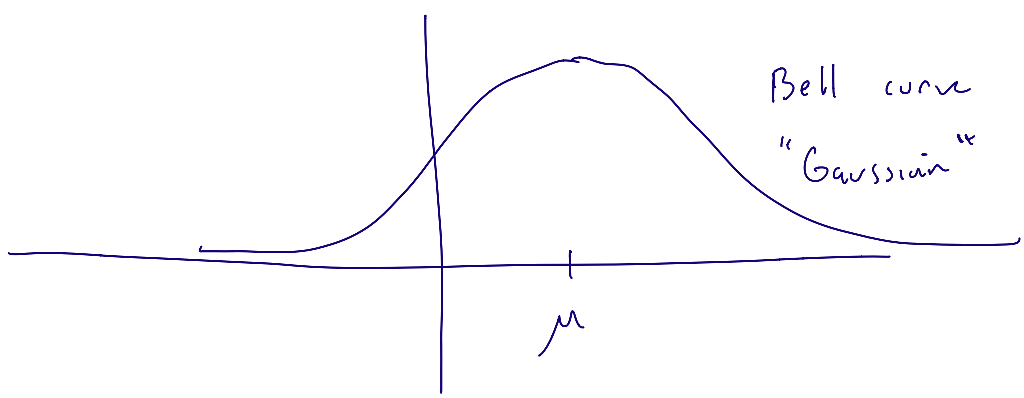
$$\text{Var}[U] = \int_0^1 \left(u - \tfrac{1}{2}\right)^2 du$$

$$= \frac{1}{12}$$

Normal Distribution

The density function of a normal random variable is:

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad \text{for } x \in (-\infty, \infty)$$



μ - controls "center" of the distribution

σ^2 - controls the "spread".

Show at home:

① $$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} \, dx = 1$$

② $$E[X] = \mu$$

③ $$Var[X] = \sigma^2$$

Example of Use

If $S_0 =$ price of stock today

$S_1 =$ price of stock tomorrow

S_1 is often modeled as

$$S_1 = S_0 e^r$$

r is the rate of return, modeled as a Normal random variable

$$\Rightarrow \frac{S_1}{S_0} = e^r$$

$$\underbrace{\log \frac{S_1}{S_0}}_{\sim \text{Normal}(\mu, \sigma^2)} = r$$

Important properties

If $X \sim \text{Normal}(\mu, \sigma^2)$ then

$Y = aX + b \sim \text{Normal}(a\mu + b, a^2\sigma^2)$

$$P[Y \le y] = P[aX + b \le y]$$
$$= P\left[X \le \frac{y-b}{a}\right]$$
$$= F_X\left(\frac{y-b}{a}\right) = F_Y$$

$$\frac{d}{dy} P[Y \le y] = f(y)$$
$$= \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$
$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{y-b}{a} - \mu\right)^2 / 2\sigma^2}$$
$$= \underbrace{\frac{1}{\sqrt{2\pi}\, a\sigma} e^{-(y-b-\mu a)^2 / 2\sigma^2 a^2}}_{\text{the density function for a } N(b+\mu a,\, a^2\mu^2) \text{ random variable.}}$$

The cumulative distribution function:

$$\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}\, dx$$ has no closed form solution.

"Standard normal distribution"

$$Z \sim N(0,1)$$

$$\Rightarrow \quad f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt = \Phi(z).$$

If $Y = aZ + b \Rightarrow Y \sim N(b, a^2)$

$$P[Y \le y] = P[aZ + b \le y]$$

$$= P\left[Z \le \frac{y-b}{a}\right] = \Phi\left(\frac{y-b}{a}\right).$$

Vice Versa, if $X \sim N(\mu, \sigma^2)$ then we can normalize it: $Z = \dfrac{X-\mu}{\sigma}$ is a standard normal random variable.

The binomial approximation

Much like we can use a Poisson r.v. to approximate a binomial r.v. in the case where n is large, p is small, $\lambda = np \sim \Theta(1)$, we can use a Normal random variable to approximate a binomial one when n is large.

DeMoivre-Laplace Limit

If $S_n \sim \text{binomial}(n, p)$, then

$$\lim_{n \to \infty} P\left[a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right] = \Phi(b) - \Phi(a)$$

$$= \int_a^b f(x)\,dx$$

std. normal density.