# Theory of Probability

Nov 2, 2020

## DeMoivre-Laplace Limit

If $S_n \sim \text{Binomial}(n,p)$, then

$$\lim_{n\to\infty} P\left[a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right] = \Phi(b) - \Phi(a)$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

The normal approximation is that

$$\frac{S_n - np}{\sqrt{np(1-p)}} \sim N(0,1)$$

is "good" when $np(1-p)$ is large.

This approximation can be used to evaluate the binomial mass function.

Let $X \sim \text{binomial}(n,p)$.

$$P[X=k] = P\Big[\underbrace{k-\tfrac{1}{2} < X < k+\tfrac{1}{2}}\Big]$$

"continuity correction" used since normal r.v.'s are continuous, binomial are discrete

$$= P\left[\frac{k-\frac{1}{2}-np}{\sqrt{np(1-p)}} < \frac{X-np}{\sqrt{np(1-p)}} < \frac{k+\frac{1}{2}-np}{\sqrt{np(1-p)}}\right]$$

$$\approx \Phi\left(\frac{k+\frac{1}{2}-np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{k-\frac{1}{2}-np}{\sqrt{np(1-p)}}\right).$$

Ex: $n = 40$, $p = \frac{1}{2}$

$$P[X = 20] = .1254$$

using the normal approximation,

$$\simeq P[19.5 < X < 20.5]$$

$$\approx .1272$$

Exponential R.V.

$X \sim Exp(\lambda)$, $\lambda > 0$, if

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , \text{ if } x \geq 0 \\ 0 & , \text{ otherwise} \end{cases}$$

$$F(x) = \int_{-\infty}^{x} \lambda e^{-\lambda t} \, dt$$

$$= \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

$$E[X] = \frac{1}{\lambda}$$

$$Var[X] = \frac{1}{\lambda^2}$$

Exponential models are memoryless:

$$P[X > s+t \mid X > t] = P[X > s] \quad (t, s > 0)$$

If $X \sim \text{Exp}(\lambda)$, then

$$P[X > s+t \mid X > t] = \frac{P[X > s+t \cap X > t]}{P[X > t]}$$

$$= \frac{P[X > s+t]}{P[X > t]}$$

$$= \frac{1 - P[X \leq s+t]}{1 - P[X \leq t]}$$

$$= \frac{1 - (1 - e^{-\lambda(s+t)})}{1 - (1 - e^{-t\lambda})}$$

$$= \frac{e^{-\lambda s}\cancel{e^{-\lambda t}}}{\cancel{e^{-\lambda t}}} = e^{-\lambda s}$$

$$= 1 - (1 - e^{-\lambda s}) = 1 - P[X \leq s]$$

$$= P[X > s].$$

## Gamma Distribution

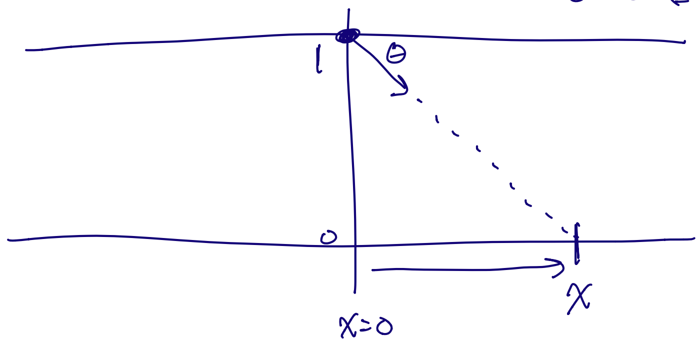
For $\alpha, \lambda > 0$,

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x} (\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1}\, dy, \quad \Gamma(n) = (n-1)!$$

# Cauchy Distribution

$\theta \in (-\pi/2, \pi/2]$




$x = \tan\theta$

$$F(x) = P[X \le x] = P[\tan\theta \le x]$$

$$= P[\theta \le \operatorname{atan} x]$$

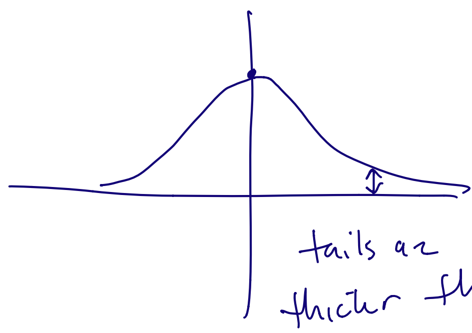
since $\theta \sim \text{Uniform}\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\operatorname{atan} x} \frac{1}{\pi}\, d\theta = \frac{\theta}{\pi}\Big|_{-\frac{\pi}{2}}^{\operatorname{atan} x} = \frac{\operatorname{atan} x}{\pi} + \frac{1}{2}$$

$$\Rightarrow f(x) = F'(x)$$

$$= \frac{d}{dx}\left(\frac{\operatorname{atan} x}{\pi} + \frac{1}{2}\right) = \frac{1}{\pi}\frac{1}{1+x^2}.$$

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\pi}\frac{1}{1+x^2}\, dx = 0$$



tails are thicker than a normal distribution.

$$Var[X] = \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{x^2}{1+x^2}\, dx$$

$$= \infty$$

The Cauchy distribution has <u>no</u> variance.