Theory of Probability

Nov 4, 2020

We say $X \sim N(\mu, \sigma^2)$ to mean that

"distributed as"

X has pdf $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$ or

that $P[X \le x] = \Phi\left(\frac{x-\mu}{\sigma}\right)$.

Question: What about $Y = g(X)$?

Answer: Compute $F_Y(y) = P[Y \le y] = P[g(X) \le y]$

Then $f_Y = F_Y'$.

Ex: $Y = X^n$ and $X \sim \text{uniform}(0,1)$.

$$\Rightarrow P[Y \le y] = P[X^n \le y] = P[X \le y^{1/n}]$$

$$= y^{1/n}.$$

$$\Rightarrow f_Y(y) = \frac{d}{dy}\left(y^{1/n}\right) = \frac{1}{n} y^{1/n - 1}.$$

Can we do the general using integration?

Consider a continuous r.v. X with pdf f_X, and g a monotonic and differentiable function.

$\Rightarrow$ g^{-1} exists everywhere,

and $\frac{d}{dy} g^{-1}$ also exists.

Let $Y = g(X)$. Then:

$$P[Y \le y] = P[g(X) \le y] = P[X \le g^{-1}(y)].$$

$$= \int_{-\infty}^{g^{-1}(y)} f_X(x)\, dx$$

$x: -\infty \to g^{-1}(y)$

Let $z = g(x)$.

$\Rightarrow z: -\infty \to y$

(assuming g is increasing)

$x = g^{-1}(z)$

$dx = \frac{d}{dz}(g^{-1}(z))\, dz$

$$= \int_{-\infty}^{y} \underbrace{f_X(g^{-1}(z))\, \frac{d}{dz}(g^{-1}(z))}\, dz$$

$$\Rightarrow \boxed{f_Y(y) = f_X(g^{-1}(y))\, \frac{d}{dy}(g^{-1}(y))}$$

If g is decreasing, be careful of signs and limits of integration.

Thm If X has pdf f_X and g strictly monotonic and differentiable, then $Y = g(X)$ has pdf

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{otherwise} \end{cases}.$$

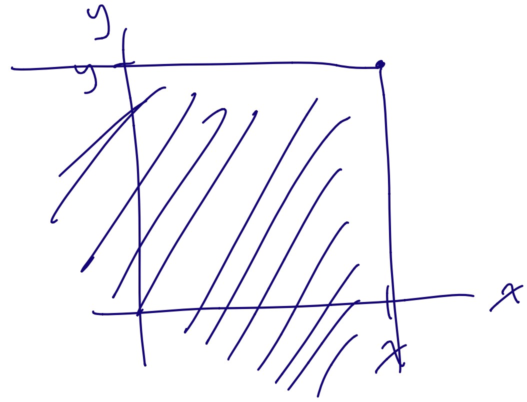
Recall $x = g^{-1}(y)$ is such that $g(x) = y$.

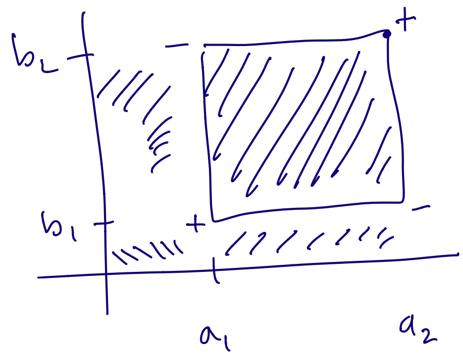
Joint Distribution Function

The Joint CDF for two r.v.s X, Y is the function F such that:

$$F(x,y) = P[X \le x, Y \le y].$$

$$P[X \in (a_1, a_2], Y \in (b_1, b_2]]$$
$$= F(a_2, b_2) - F(a_2, b_1) \underbrace{- F(a_1, b_2) + F(a_1, b_1)}_{\text{Inclusion - Exclusion.}}$$





Discrete Case

Joint prob. mass function:

$$p(x_i, y_j) = P[X = x_i, Y = y_j]$$

$$\sum_{i,j} p(x_i, y_j) = 1$$

$$p_X(x_i) = P[X = x_i]$$
$$= P\left[\bigcup_j \{X = x_i, Y = y_j\}\right]$$
$$= \sum_j p(x_i, y_j)$$

Continuous Case

$$P[\underset{\substack{\uparrow \\ \text{any region} \\ \text{in } \mathbb{R}^2}}{X, Y \in C}] = \iint_C \underbrace{f(x,y)}_{\text{joint pdf.}} \, dx \, dy$$

$$P[X \in (a,b), Y \in (c,d)] = \int_c^d \int_a^b f(x,y) \, dx \, dy.$$

The distribution function is then

$$F(x,y) = P[X \le x, Y \le y]$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f(u,v)\, du\, dv$$

And since we have this formula,

$$\frac{\partial^2 F}{\partial x \partial y} = f.$$

And lastly $$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx.$$

Everything extends to the n-dimensional case of n random variables $X_1, X_2, \ldots X_n$.