Theory of Probability

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X, Y are independent random variables if for any two sets of real numbers A, B:

$$P[X \in A, Y \in B] = P[X \in A] \cdot P[Y \in B]$$

$$\iff F(a,b) = \text{joint CDF} \qquad (P[X \le a, Y \le b])$$
$$= F_X(a)\, F_Y(b)$$

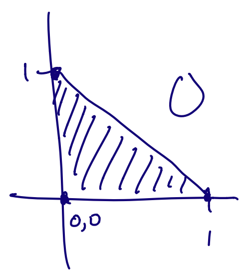
$$\iff f(x,y) = \text{joint PDF} \qquad \text{[See Proposition 2.1]}$$
$$= f_X(x) \cdot f_Y(y)$$

Ex: $f(x,y) = 6 e^{-2x} e^{-3y}, \quad x, y > 0$

$$\Rightarrow f(x,y) = \underbrace{2e^{-2x}}_{f_X} \cdot \underbrace{3e^{-3y}}_{f_Y}$$

Ex: Let X, Y have pdf $f(x,y) = \begin{cases} 24xy & \text{for } x \in 0,1 \\ & \quad y \in 0,1 \\ & \quad x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$

This pdf cannot be separated into the product of a function of X and a function of y.



$f(x,y) = 24xy\, \mathbb{1}$ ← is a function of $x+y$, not xy.

↑ indicator function

Sums of Independent Random Variables

Question: If X and Y are independent, then let $Z = X + Y$. How is Z distributed?

(I.e. what is f_Z or F_Z?)

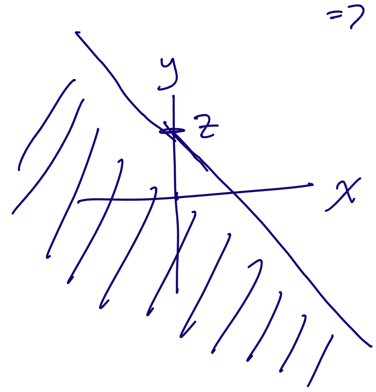
$$F_Z(z) = P[Z \le z]$$

$$= P[\underbrace{X+Y} \le z]$$

$$= \iint_{x+y \le z} f(x,y)\, dx\, dy$$

$x + y \le z$

$\Rightarrow y \le z - x$



$$= \iint_{x+y \le z} f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} F_X(z-y)\, f_Y(y)\, dy$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} \int_{-\infty}^{\infty} F_X(z-y)\, f_Y(y)\, dy$$

$$= \underbrace{\int_{-\infty}^{\infty} f_X(z-y)\, f_Y(y)\, dy}_{\text{Convolution of } f_X \text{ with } f_Y}$$

$$= f_X * f_Y$$

IID = Independent Identically Distributed

Consider 2 IID $U(0,1)$ random variables X, Y.

$$Z = X + Y$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)\, f_Y(y)\, dy$$

$$= \int_0^1 f_X(z-y)\, dy$$

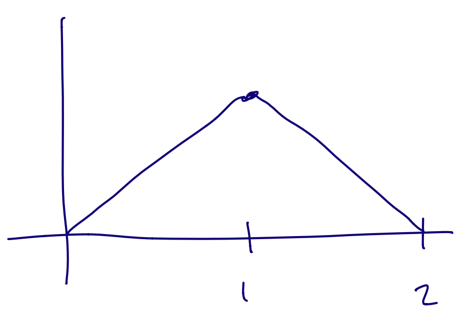
If $z \in (0,1)$ $\Rightarrow$ $f_X(z-y)$ is non-zero when

$$\Rightarrow \int_0^z 1\, dy = z$$

If $z \in (1,2)$ $\Rightarrow$ $f_X(z-y)$ is non-zero when

$$\Rightarrow \int_{z-1}^1 1\, dy = y\Big|_{z-1}^{1} = 2 - z$$

$$\Rightarrow \quad f_Z(z) = \begin{cases} z & \text{on } z \in (0,1) \\ 2-z & \text{on } z \in [1,2) \\ 0 & \text{otherwise} \end{cases}$$



Prop: $X \sim \Gamma(s, \lambda)$

$Y \sim \Gamma(t, \lambda)$

then $Z = X + Y \sim \Gamma(s+t, \lambda)$.

Prop: If $X_1 \sim N(\mu_1, \sigma_1^2)$

$\vdots$

$X_n \sim N(\mu_n, \sigma_n^2)$

then $Y = \sum_{i=1}^{n} X_i \sim N\left(\sum_j \mu_j, \sum_j \sigma_j^2\right)$

In the discrete case

Let X take values $0, 1, 2, \ldots$

Y take values $0, 1, 2 \ldots$

$Z = X + Y$, then Z takes values $0, 1, \ldots$

$$P[Z=n] = \sum_{k=0}^{n} P[X=k, Y=n-k]$$

$$= \underbrace{\sum_{k=0}^{n} P[X=k]\,P[Y=n-k]}_{\text{discrete convolution.}}$$

Ex: $X \sim \text{Poision}(\lambda_1)$

$Y \sim \text{Poision}(\lambda_2)$

$$P[X+Y=n] = \sum_{k=0}^{n} \frac{\lambda_1^k e^{-\lambda_1}}{k!} \frac{\lambda_2^{n-k} e^{-\lambda_2}}{(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} n! \frac{\lambda_1^k \lambda_2^{n-k}}{k!(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k}$$

$$= \underbrace{\frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1+\lambda_2)^n}_{\text{probability mass function for Poision}(\lambda_1+\lambda_2).}$$ by the Binomial Thm.

$\Rightarrow \quad X+Y \sim \text{Poision}(\lambda_1+\lambda_2).$