Theory of Probability   Nov 16, 2020

Conditional Distributions

Goal: Given X, Y random variables with joint pdf $f(x,y)$ and marginal pdfs f_X, f_Y, what is the pdf of the random variable

$$Z = \underbrace{X \mid Y = y}_{\text{"X given Y"}}$$

First: Discrete Case

Recall for events:

$$P[E \mid F] = \frac{P[EF]}{P[F]}$$

Define the <u>conditional probability mass function</u> so that

$$p_{X|Y}(x|y) = P[X = x \mid Y = y]$$

$$= \frac{P[X = x, Y = y]}{P[Y = y]}$$

$$= \frac{p(x,y)}{p_Y(y)}$$

joint mass function

marginal mass function.

(assuming $p_Y(y) > 0$).

Conditional Distribution function is then

$$F_{X|Y}(x|y) = P[X \le x \mid Y = y]$$

$$= \sum_{a \le x} P[X = a \mid Y = y]$$

$Y \backslash X$	x_1	x_2	x_3	$\dots$
y_1				
y_2		$p(x_2, y_2)$		
y_3				
$\vdots$				

Grid of probabilities

Fix $Y = y_3$,

scale this row by $1/p_Y(y_3)$ so that

$$\sum_i p_{X|Y}(x_i \mid y_3) = 1.$$

If X, Y are independent random variables,

then $\quad p(x,y) = p_X(x)\, p_Y(y).$

$$\Rightarrow \quad p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}$$

$$= \frac{p_X(x)\, \cancel{p_Y(y)}}{\cancel{p_Y(y)}}$$

$$= p_X(x).$$

Continuous Case

Define $$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

Motivation: multiply each side by $dx = \frac{dx\,dy}{dy}$

$$\Rightarrow \quad f_{X|Y}(x|y)\,dx = \frac{f(x,y)\,dx\,dy}{f_Y(y)\,dy}$$

$$\approx \frac{P[X \in (x, x+dx), Y \in (y, y+dy)]}{P[Y \in (y, y+dy)]}.$$

$$\Rightarrow \quad P[X \in A \mid Y = y] = \int_A f_{X|Y}(x|y)\,dx$$

likewise $$P[X \in A \mid Y \in B] = \int_A \int_B f_{X|Y}(x|y)\,dy\,dx.$$

Distribution function:

$$A = (-\infty, x)$$

$$\Rightarrow \quad F_{X|Y}(x|y) = P[X \leq x \mid Y = y]$$

$$= \int_{-\infty}^{x} f_{X|Y}(u|y)\,du.$$

If X, Y are independent, then

$$f_{X|Y}(x|y) = f_X(x).$$