Theory of Probability

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Recall:

Discrete $E[X] = \sum_i x_i \, p(x_i)$

Continuous $E[X] = \int x f(x) \, dx$

If X, Y have joint pdf $f(x,y)$, then

$$E[g(X,Y)] = \iint g(x,y) f(x,y) \, dy .$$

Ex: If $g(x,y) = x+y$.

$$\Rightarrow E[g(X,Y)] = \iint (x+y) f(x,y) \, dx\,dy$$

$$= \iint x f(x,y) \, dx\,dy + \iint y f(x,y) \, dx\,dy$$

$$= \int x f_X(x) \, dx + \int y f_Y(y) \, dy$$

$$= E[X] + E[Y].$$

$$\Rightarrow E[X_1 + \dots + X_n] = \sum_{i=1}^{n} E[X_i].$$

Beware: It is not necessarily the case that

$$E\left[\sum_{i=1}^{\infty} X_i\right] = \sum_{i=1}^{\infty} E[X_i] \, !$$

$$E\left[\sum_{i=1}^{\infty} X_i\right] = E\left[\lim_{n\to\infty} \sum_{i=1}^{n} X_i\right]$$

$$\overset{?}{=} \lim_{n\to\infty} E\left[\sum_{i=1}^{n} X_i\right]$$

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Two cases in which the limit can be interchanged:

1. $X_i \geq 0$
2. $\sum_{i=1}^{\infty} E[|X_i|] < \infty$ "absolutely convergent"

Covariance, Variance of Sums, Correlations

Covariance of X and Y is defined as:

$$\mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

where $\mu_X = E[X]$

$\mu_Y = E[Y]$.

multiplying out $= E[XY] - E[X]\,E[Y]$.

$$\iint xy\, f(x,y)\, dx\, dy.$$

If X, Y are independent, then $E(XY) = E[X]\,E[Y]$.

$$\mathrm{Cov}(X,Y) = E[X]\,E[Y] - E[X]\,E[Y] = 0$$

Properties of Covariance

① $\text{Cov}(X,Y) = \text{Cov}(Y,X)$

② $\text{Cov}(X,X) = \text{Var}(X).$

③ $\text{Cov}(aX, Y) = a\,\text{Cov}(X,Y)$

④ $\text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \text{cov}(X_i, Y_j)$

$$\Rightarrow \text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \text{cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} \text{Var}(X_i) + \sum_{i \neq j} \text{cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i<j} \text{cov}(X_i, X_j).$$

If the X_i's are independent, then

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i).$$

Correlation:

The correlation of X, Y is:

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}}$$

Going back to linear algebra, consider the inner product (dot product):

$$(\vec{x},\vec{y}) = \sum_{i=1}^{n} x_i y_i \quad \Big\} \text{ covariance}$$

$$\left.\begin{aligned} \|\vec{x}\|^2 &= (\vec{x},\vec{x}) \\ &= \sum_{i=1}^{n} x_i^2 \end{aligned}\right\} \text{ variance}$$

angle between $\vec{x}, \vec{y}$

$$\underbrace{(\vec{x},\vec{y})}_{\text{"covariance"}} = \underbrace{\|\vec{x}\|}_{\text{"stdev"}} \underbrace{\|\vec{y}\|}_{\text{"stdev"}} \underbrace{\cos\theta_{xy}}_{\rho}$$

$$\Rightarrow \quad \cos\theta_{xy} = \frac{(\vec{x},\vec{y})}{\|\vec{x}\| \, \|\vec{y}\|}$$

analogous formulas.

compare with correlation:

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}}$$

$\rho(X,Y) = 0 \quad \Rightarrow \quad$ uncorrelated (orthogonal)

$\rho(X,Y) = 1 \quad \Rightarrow \quad X = aY + b$ (X, Y colinear).