Theory of Probability

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Strong Law of Large Numbers (SLLN)

Let $X_1, X_2, \ldots$ be a sequence of IID random variables with $E[X_i] = \mu < \infty$. Then

$$P\left[\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = \mu\right] = 1.$$

Weak Law of Large Numbers (WLLN)

For any $\epsilon > 0$, $\lim_{n\to\infty} P\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \epsilon\right] = 0.$

Differences: Additional assumptions in the proof in the text:

$$\text{WLLN}: \quad Var[X_i] = \sigma^2 < \infty$$

$$\text{SLLN}: \quad E[X_i^4] < \infty.$$

Furthermore:

$$\text{SLLN} \Rightarrow \text{WLLN}.$$

Additional Inequalities

We we may be interested in estimating

$P[X - \mu \geq a]$, when only $E[X] = \mu$ and $Var[X] = \sigma^2$ are known. (for $a > 0$)

Trivially, since $X-\mu \geq a \Rightarrow |X-\mu| \geq a$,

we can immediately apply Chebyshev's Inequality:

$$P[X-\mu \geq a] \leq P[|X-\mu| \geq a] \leq \frac{\sigma^2}{a^2} \quad \text{for } a>0.$$

<u>Proposition</u> One-sided Chebyshev Inequality:

If $E[X]=0$, $P[X \geq a] \leq \dfrac{\sigma^2}{\sigma^2+a^2} \leq \dfrac{\sigma^2}{a^2}$.

<u>Proof:</u> Let $b>0$ and note that

$X \geq a$ is equivalent to $X+b \geq a+b$,

$$\Rightarrow P[X \geq a] = P[X+b \geq a+b]$$
$$\leq P[(X+b)^2 \geq (a+b)^2]$$
$$\leq \frac{E[(X+b)^2]}{(a+b)^2} = \frac{\sigma^2+b^2}{(a+b)^2}$$

Set $b = \sigma^2/a$ (which minimizes $\dfrac{\sigma^2+b^2}{(a+b)^2}$)

$$\Rightarrow P[X \geq a] \leq \frac{\sigma^2}{\sigma^2+a^2}.$$

<u>Jensen's Inequality</u>

If f is a convex function, i.e., $f''(x) \geq 0$ for all x,

then $E[f(X)] \geq f(E[X])$,

assuming $E[f(X)]$, $E[X]$ exist and are finite

<u>Proof</u>:

Write $f(x) = \overset{\mu = E[X]}{f(\mu)} + f'(\mu)(x-\mu) + \dfrac{f''(\xi)(x-\mu)^2}{2}$ $\quad \xi \in [x, \mu]$

Since $f''(\xi) \geq 0$,

$$f(x) \geq f(\mu) + f'(\mu)(x-\mu)$$

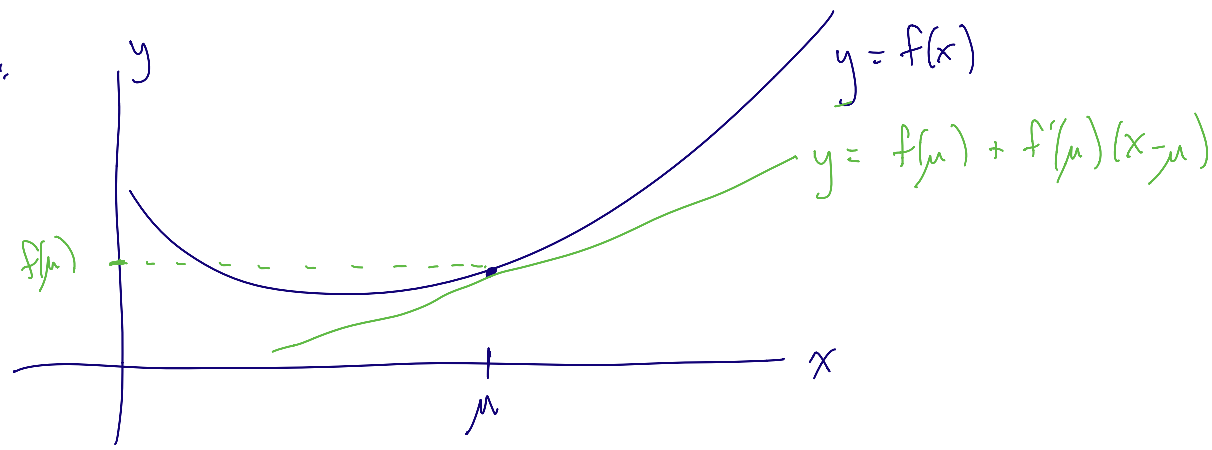
$$\Rightarrow \quad f(X) \geq f(\mu) + f'(\mu)(X-\mu)$$

$$\Rightarrow \quad E[f(X)] \geq E[f(\mu) + f'(\mu)(X-\mu)]$$

$$= f(\mu) + f'(\mu)\underbrace{E[X-\mu]}_{=0}$$

$$= f(E[X]). \checkmark$$

<u>Idea</u>:



The line $y = f(\mu) + f'(\mu)(x-\mu)$ is always below the curve $y = f(x)$

$$\Rightarrow \quad f(x) \geq f(\mu) + f'(\mu)(x-\mu).$$