Theory of Probability   Sep 23, 2020

Conditional probability: $P(E|F) = \frac{P(EF)}{P(F)}$

$P(EF) = P(E|F)\,P(F)$

Multiplication Rule

$P(ABC) = P(A|BC)\,P(BC)$

$= P(A|BC)\,P(B|C)\,P(C)$.

$P(E_1E_2\cdots E_n) = P(E_1)\,P(E_2|E_1)\,P(E_3|E_1E_2)\ldots P(E_n|E_1\cdots E_{n-1})$

Bayes Rule

$P(E|F) = \frac{P(EF)}{P(F)}$ $P(F|E) = \frac{P(EF)}{P(E)}$.

$P(EF) = P(E|F)\,P(F)$ $P(EF) = P(F|E)\,P(E)$

$\Rightarrow \quad P(E|F)\,P(F) = P(F|E)\,P(E)$

$$P(E|F) = \frac{P(F|E)\,P(E)}{P(F)}$$

Law of total probability

If $F_1, \ldots, F_n$ are mutually exclusive and $\bigcup_i F_i = S$,

and therefore for any E,

$$P(E) = \sum_i \underline{P(EF_i)}$$

$$= \sum_i P(E|F_i)\,P(F_i) \longleftarrow \text{most useful.}$$

$$= \sum_i P(F_i|E)\,P(E)$$

Problem 3.51

C = event male has cancer

T = event that PSA level is elevated.

(a) $$P(C|T) = \frac{P(T|C)\,P(C)}{P(T)}$$

$P(C) = .70$

$P(T|C) = .268$

$$P(T) = \underbrace{P(T|C)P(C) + P(T|C^c)\,P(C^c)}_{\text{Law of total probability}}$$

$$= (.268)(.70) + (.135)(.30)$$

$$= .2281$$

$$P(C|T) = \frac{(.268)(.70)}{.2281} = .8224$$

(b) $$P(C|T^c) = \frac{P(T^c|C)\,P(C)}{P(T^c)}$$

$$= \frac{P(T^c|C)\,P(C)}{P(T^c|C)P(C) + P(T^c|C^c)\,P(C^c)}$$

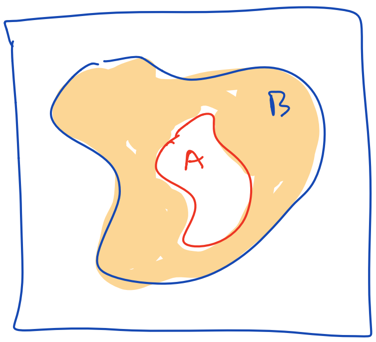
$$= \frac{P(T^c|C)\,P(C)}{1 - P(T)}$$

$$= \frac{(.732)(.70)}{.7719} = .6638$$

$$1 - \underbrace{P(T \mid C)}_{.268} = P(T^c \mid C)$$

Theoretical Exercise 3.2

Let $A \subset B$.



① $P(A \mid B) = \dfrac{P(AB)}{P(B)} = \dfrac{P(A)}{P(B)}$

② $P(A \mid B^c) = \dfrac{P(AB^c)}{P(B)} = 0.$

③ $P(B \mid A) = 1$

④ $P(B \mid A^c) = \dfrac{P(BA^c)}{P(A^c)} = \dfrac{P(A^c \mid B)\,P(B)}{P(A^c)}$

$$= \frac{P(B) - P(A)}{P(A^c)}$$

Theoretical Exercise 3.5

(a) Let E, F be mutually exclusive events.

Show that $P(E \mid E \cup F) = \dfrac{P(E)}{P(E) + P(F)}$.

$$P(E \mid E \cup F) = \frac{P(E \cap (E \cup F))}{P(E \cup F)}$$

$$= \frac{P(E)}{P(E) + P(F)}$$

(b) If E_i are mutually exclusive, then show

$$P\left(E_j \mid \bigcup_i E_i\right) = \frac{P(E_j)}{\sum_i P(E_i)}.$$

$$= \frac{P\left(E_j \cap \left(\bigcup_i E_i\right)\right)}{P\left(\bigcup_i E_i\right)} = \frac{P(E_j)}{\sum_i P(E_i)}.$$