

# Theory of Probability

Sep 28, 2020

Events  $E$  &  $F$  are independent if

$$P(EF) = P(E)P(F)$$

Prop If  $\bar{E}, \bar{F}$  are independent then so are the two events  $E$  and  $F^c$ .

If we consider the multiple events  $E_1, \dots, E_n$ , this set of events are independent if for any subset  $E_{i_1}, \dots, E_{i_r}$ , with  $r \leq n$ , we have that

$$P(E_{i_1} E_{i_2} \dots E_{i_r}) = P(E_{i_1}) \cdot P(E_{i_2}) \dots P(E_{i_r}).$$

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## Example 4F

Independent trials: a sequence of experiments, each of whose outcome is independent from the others.

Imagine an infinite sequence of trials, each trial has probability of success  $p$ , and failure  $1-p=q$ .

(a) What is the probability that there is at least one success in the first  $n$  trials?

$$\begin{aligned} \Rightarrow 1 - P(F_1 F_2 F_3 \dots F_n) &= 1 - P(F_1)P(F_2) \dots P(F_n) \\ &= 1 - (1-p)^n. \end{aligned}$$

(b) What is the probability of exactly  $k$  successes in the first  $n$  trials?

F F S F S S ... F S F

n trials

$$P(\text{of a particular arrangement of } k \text{ successes}) = p^k (1-p)^{n-k}$$

# of such arrangements of  $k$  successes is  $\binom{n}{k}$ .

$$\Rightarrow P(\text{exactly } k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}.$$

(c) Probability that all trials are successes?

$$P(n \text{ success in } n \text{ trials}) = p^n.$$

$$P(\text{all success}) = \lim_{n \rightarrow \infty} p^n = \begin{cases} 0 & \text{if } 0 \leq p < 1 \\ 1 & \text{if } p = 1 \end{cases}$$

### Example 45

Probability of success is  $p$ , failure  $1-p$ .

What is the probability of  $n$  successes before  $m$  failures.

Pascal's Solution:

Consider the first trial: Let  $E$  = event that we get  $n$  successes before  $m$  failures.

We are interested in

$$P(E) = P_{n,m}$$

$$= P(EF) + P(EF^c)$$

where  $F$  = event trial 1 is a success

$$= P(\text{Success on trial one} \cap \text{n-1 subsequent successes before m failures}) + P(\text{failure on trial 1} \cap \text{n successes before m-1 failures})$$

$$= p P_{n-1,m} + (1-p) P_{n,m-1}$$

$$\Rightarrow P_{n,m} = p P_{n-1,m} + (1-p) P_{n,m-1} \quad \text{for } n \geq 1, m \geq 1.$$

Recurrence relation ↗

We also know the boundary conditions:

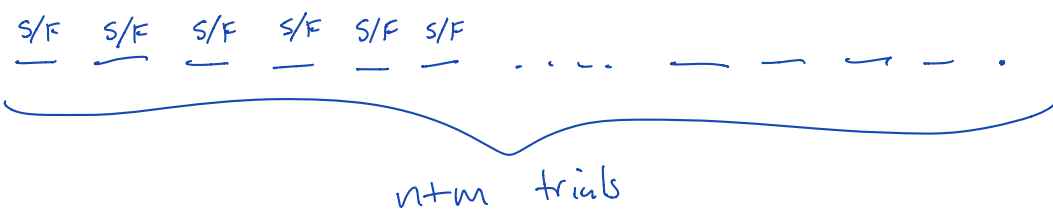
$$P_{n,0} = 0, \quad P_{0,m} = 1$$

$$P_{1,m} = p P_{0,m} + (1-p) P_{1,m-1}$$

$$P_{n,1} = p P_{n,0} + (1-p) P_{n,1}$$

Fermat's Solution

If  $\underline{n}$  successes occur before  $\underline{m}$  failures:



$\geq n$  successes  
or  $\geq m$  failures

$\Rightarrow \geq n$  successes in the first  $n+m-1$  trials.

Before we get to the last trial, we had to have lost  $\geq n$  successes, otherwise we've had  $m$  failures and lost.

$$P(k \text{ successes in } n+m-1 \text{ trials}) = \binom{m+n-1}{k} p^k (1-p)^{m+n-1-k}$$

$$P_{n,m} = \sum_{k=n}^{m+n-1} \binom{m+n-1}{k} p^k (1-p)^{m+n-1-k} \quad \text{since}$$

the event of winning with  $k$  successes is mutually exclusive from the event of winning with  $k'$  successes if  $k \neq k'$ .

Solving the recurrence relationship:

$$P_{n,m} = p P_{n-1,m} + (1-p) P_{n,m-1} \quad n \geq 1, m \geq 1$$

linear equation in 3 unknowns:  $n \leq 10, m \leq 10$

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -(1-p) & -p & 1 & & \end{pmatrix} \begin{pmatrix} P_{1,0} \\ P_{0,1} \\ P_{1,1} \\ P_{2,0} \\ P_{0,2} \\ \vdots \\ P_{10,10} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}$$