Theory of Probability

Sep 28, 2020

Events E & F are <u>independent</u> if

$$P(EF) = P(E)\,P(F)$$

Prop If E, F are independent then so are the two events E and F^c.

If we consider the multiple events $E_1, \dots, E_n$, this set of events are independent if for any subset $E_{i_1}, \dots, E_{i_r}$, with $r \leq n$, we have that

$$P(E_{i_1} E_{i_2} \cdots E_{i_r}) = P(E_{i_1}) \cdot P(E_{i_2}) \cdots P(E_{i_r}).$$

Example 4F

Independent trials : a sequence of experiments, each of whose outcome is independent from the others.

Imagine an infinite sequence of trials, each trial has probability of success p, and failure $1-p = q$.

(a) What is the probability that there is <u>at least</u> one success in the first n trials?

$$\Rightarrow 1 - P(F_1 F_2 F_3 \cdots F_n) = 1 - P(F_1)P(F_2) \cdots P(F_n)$$

$$= 1 - (1-p)^n.$$

(b) What is the probability of exactly k successes in the first n trials?

$$\underbrace{\underline{F}\ \underline{F}\ \underline{S}\ \underline{F}\ \underline{S}\ \underline{S}\ \cdots\ \underline{F}\ \underline{S}\ \underline{F}}_{n \text{ trials}}$$

$$P\begin{pmatrix}\text{of a particular} \\ \text{arrangement of} \\ k \text{ successes}\end{pmatrix} = p^k(1-p)^{n-k}$$

# of such arrangements of k successes is $\binom{n}{k}$.

$$\Rightarrow \quad P(\text{exactly } k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}.$$

(c) Probability that all trials are successes?

$$P(n \text{ success in } n \text{ trials}) = p^n.$$

$$P(\text{\underline{all} success}) = \lim_{n\to\infty} p^n = \begin{cases} 0 & \text{if } 0 \le p < 1 \\ 1 & \text{if } p = 1 \end{cases}$$

<u>Example 4J</u>

Probability of success is p, failure $1-p$.

What is the probability of n successes <u>before</u> m failures.

<u>Pascal's Solution</u>:

Consider the first trial: Let E = event that we get n successes before m failures.

We are interested in

$$P(E) = P_{n,m}$$

$$= P(EF) + P(EF^c)$$

where F = event trial 1 is a success

$$= P\left(\text{Success on trial one} \cap \begin{array}{c}\text{n-1 subsequent}\\ \text{successes before}\\ \text{m failures}\end{array}\right) + P\left(\begin{array}{c}\text{failure on}\\ \text{trial 1}\end{array} \cap \begin{array}{c}\text{n successes}\\ \text{before m-1}\\ \text{failures}\end{array}\right)$$

$$= p\, P_{n-1,m} + (1-p)\, P_{n,m-1}$$

$$\Rightarrow P_{n,m} = p\, P_{n-1,m} + (1-p)\, P_{n,m-1} \quad \text{for } n \geq 1,\ m \geq 1.$$

Recurrence relation

We also know the boundary conditions:

$$P_{n,0} = 0 \quad , \quad P_{0,m} = 1$$

$$P_{1,m} = p\, P_{0,m} + (1-p)\, P_{1,m-1}$$

$$P_{1,1} = p\, P_{0,1} + (1-p)\, P_{1,0}$$

Fermat Solution

If $\underline{n}$ successes occur before $\underline{m}$ failures:

$$\underbrace{\underline{S/F}\ \ \underline{S/F}\ \ \underline{S/F}\ \ \underline{S/F}\ \ \underline{S/F}\ \ \underline{S/F}\ \ \ldots\ \ \_\ \ \_\ \ \_\ \ \_\ .}_{n+m \text{ trials}}$$

$\geq n$ successes or $\geq m$ failures

$\Rightarrow$ $\geq n$ successes in the first $n+m-1$ trials.

Before we get to the last trial, we had to have

had $\geq n$ successes, otherwise we'd had m failures and lost.

$$P\left(\begin{matrix}k \text{ successes in} \\ n+m-1 \text{ trials}\end{matrix}\right) = \binom{m+n-1}{k} p^k (1-p)^{m+n-1-k}$$

$$P_{n,m} = \sum_{k=n}^{m+n-1} \binom{m+n-1}{k} p^k (1-p)^{m+n-1-k}$$ since

the event of winning with k successes is mutually exclusive from the event of winning with k' successes if $k \neq k'$.

Solving the recurrence relationship:

$$\underbrace{P_{n,m} = p P_{n-1,m} + (1-p) P_{n,m-1}}_{\text{linear equation in 3 unknowns.}} \qquad n \geq 1, \ m \geq 1$$

$n \leq 10$

$m \leq 10$

$$\begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ -(1-p) & -p & 1 & & \\ & & & & \\ & & & & \end{pmatrix} \begin{pmatrix} P_{1,0} \\ P_{0,1} \\ P_{1,1} \\ P_{2,0} \\ P_{0,2} \\ P_{\cdot} \\ \vdots \\ P_{10,10} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ \vdots \end{pmatrix}$$