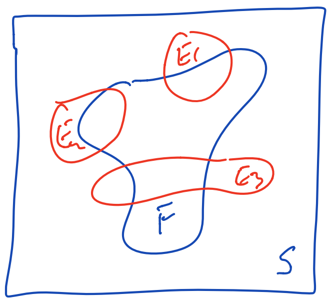
Theory of Probability

Call $Q(E) = P(E|F)$ (for a fixed F), then

Q satisfies the axioms of probability.



$$P(E_1|F) = \frac{P(E_1 F)}{P(F)}$$

By conditioning on F, we are effectively "redefining" the sample space S to be just F.

$$Q(E) = P(E|F)$$

$$Q(E|G) = \frac{Q(EG)}{Q(G)} = \frac{P(EG|F)}{P(G|F)} = \frac{P(EGF)}{\cancel{P(F)}} \frac{\cancel{P(F)}}{P(GF)}$$

$$= \frac{P(EGF)}{P(GF)} = P(E|GF).$$

Conditional Independence

E_1, E_2 are conditionally independent with respect to F if $P(E_1|E_2F) = P(E_1|F)$

this is equivalent to $P(E_1E_2|F) = P(E_1|F)\, P(E_2|F)$

$\iff Q(E_1E_2) = Q(E_1)\, Q(E_2)$.

Ex 3.33 (self test)

Let E, F, G be independent events:

Show that $P(E \mid FG^c) = P(E)$

$$P(E \mid FG^c) = \frac{P(EFG^c)}{P(FG^c)} = \frac{P(E)\,\cancel{P(F)}\,\cancel{P(G^c)}}{\cancel{P(F)}\,\cancel{P(G^c)}} = P(E).$$

Ex 3.35 (self test)

An urn contains 4 red, 5 white, 6 blue, and 7 green balls. Randomly choose 4 balls (without replacement)

$$P(\underbrace{\text{all white balls}}_{\text{white}} \mid \underbrace{\text{all ball are same color}}_{\text{same}}) \; ?$$

$$= \frac{P(\text{white AND same})}{P(\text{same})} = \frac{P(\text{white})}{P(\text{same})}$$

$$P(\text{white}) = \frac{\binom{5}{4}}{\binom{22}{4}} \qquad P(\text{same}) = \frac{1 + \binom{5}{4} + \binom{6}{4} + \binom{7}{4}}{\binom{22}{4}}$$

$$P(\text{white} \mid \text{same}) = \frac{\binom{5}{4}}{1 + \binom{5}{4} + \binom{6}{4} + \binom{7}{4}} = \frac{5}{1+5+15+35} = \frac{5}{56}$$

Ex 3.27 (self test)

An urn contains 1R and 1B ball.

Question: At stage n, show that the probability that urn contains i red balls is $\frac{1}{n+1}$, for $i = 1, \dots, n+1$.

Stage 0: 1R, 1B

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Stage 1: 2R, 1B or 1R, 2B

2/3 1/3 1/3 2/3

Stage 2: 3R, 1B 2R, 2B 2R, 2B 1R, 3B

$\frac{1}{3}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{3}$

3R, 1B	1/3
2R, 2B	1/3
1R, 3B	1/3

Use Mathematical Induction.

1. Showed it was true for $n = 0, 1, 2$
2. Assume the result holds after stage n.
3. Assuming 2, show the result is true for stage $n+1$.

Compute $P\left(\begin{matrix} i \text{ red balls} \\ \text{stage } n+1 \end{matrix}\right)$

$\frac{1}{n+1}$

$$= P\left(\begin{matrix} i \text{ red balls} \\ \text{stage } n+1 \end{matrix} \,\middle|\, \begin{matrix} i \text{ red balls} \\ \text{stage } n \end{matrix}\right) P\left(\begin{matrix} i \text{ red balls} \\ \text{stage } n \end{matrix}\right)$$

$$+ P\left(\begin{matrix} i \text{ red balls} \\ \text{stage } n+1 \end{matrix} \,\middle|\, \begin{matrix} i-1 \text{ red balls} \\ \text{stage } n \end{matrix}\right) P\left(\begin{matrix} i-1 \text{ red balls} \\ \text{stage } n \end{matrix}\right)$$

$\frac{1}{n+1}$

number of blue balls

$$= \frac{n+2-i}{n+2} \cdot \frac{1}{n+1} + \frac{i-1}{n+2} \cdot \frac{1}{n+1}$$

$$= \frac{n+2 \cancel{-i} \cancel{+i} -1}{(n+2)(n+1)} = \frac{n+1}{(n+2)(n+1)} = \frac{1}{n+2}. \quad \blacksquare$$

If $F_1, \ldots, F_n$ are mutually exclusive, and $\bigcup F_j = S$

$$P(E) = \sum_j P(EF_j)$$

$$= \sum_j P(E \mid F_j) P(F_j).$$

$$P(E \mid F_j) = \frac{P(EF_j)}{P(F_j)}$$

$$\Rightarrow P(EF_j) = P(E \mid F_j) P(F_j)$$

This idea is often used in conjunction with Bayes's Theorem.