Theory of Probability

Oct 5, 2020

Random Variables

Discrete random variables :

X is a discrete random var. if it can take on a countable set of values $x_1, x_2, \ldots \to$

Probability mass function (density) $\quad p(x_i) = P(X = x_i)$

(Cumulative) Distribution function

$$F(x) = P(X \le x)$$

$$= \sum_{x_i \le x} P(X = x_i)$$

$$= \sum_{x_i \le x} p(x_i)$$

- F is <u>non-decreasing</u>

$\Rightarrow \quad F(y) \ge F(x) \quad$ if $\quad y > x$

Example: Die $\quad P(X = 1) = \frac{1}{6}$

$\vdots$

$P(X = 6) = 1/6$

$F(3) = P(X \le 3) = \frac{1}{2}$

$F(3.5) = P(X \le 3.5) = 1/2$.

Expected value:

$$E(X) = \sum_{i=1}^{\infty} x_i \cdot P(X = x_i)$$
$$= \sum_{i=1}^{\infty} x_i \, p(x_i)$$

Theoretical Ex: 4.2

If X has distribution function F, $P(X \le x) = F(x)$, then what is the distribution function of e^X?

$\Rightarrow$ $Y = e^X$ is another random variable

$$G(y) = P(Y \le y)$$
$$= P(e^X \le y)$$
$$= \underbrace{P(X \le \log y)}_{F(\log y)} = F(\log y)$$

$\Rightarrow$ $e^X = Y$ has distribution function $G(y) = F(\log y)$.

Self-test Ex: 4.3

A coin comes up heads with probability p.
Flip this coin until either heads or tails has occurred twice. Find the expected number of flips.

Let X = the number of flips until 2 heads or 2 tails.

$$P(X=0) = 0$$
$$P(X=1) = 0$$
$$P(X=2) = P(HH) + P(TT)$$
$$= p^2 + (1-p)^2$$

$$P(X=3) = P(HTT) + P(HTH) + P(THH) + P(THT)$$
$$= 1 - P(X=2) = 1 - p^2 - (1-p)^2$$

$$P(X \geq 4) = 0$$

$$E(X) = 2 \cdot P(X=2) + 3 \cdot P(X=3)$$
$$= 2\left(p^2 + (1-p)^2\right) + 3\left(1 - p^2 - (1-p)^2\right)$$
$$= 3 - p^2 - (1-p)^2 .$$

If $p = 0$, $E(X) = 3 - 0 - (1-0)^2 = 2$

$p = 1$, $E(X) = 3 - 1 - 0 = 2$

$p = \frac{1}{2}$, $E(X) = 3 - \frac{1}{4} - \left(\frac{1}{4}\right) = 2.5$

Self test 4.4

Suppose we have M families, n_i of which have i children, $\sum_{i=1}^{r} n_i = M$.

Let X = the number of children in a randomly selected family.

Also consider, instead of picking a family at random, pick a child at random. Then are

$$C = \sum_{i=1}^{r} n_i \cdot i \,.$$

Let Y = the number of children in the family of the randomly selected child.

Goal: Show that $E(Y) \geq E(X)$.

$$P(X = i) = P\left(\text{choose a family with } i \text{ children}\right) = \frac{n_i}{M}.$$

$$E(X) = \sum_{i=1}^{r} i \cdot \underbrace{\frac{n_i}{M}}_{P(X=i)}$$

$$P(Y = i) = ?$$

$$= \frac{i \cdot n_i}{C}$$

$$E(Y) = \sum_{i=1}^{r} i \cdot \frac{n_i \cdot i}{C} = \sum_{i=1}^{r} \frac{i^2 n_i}{C}$$

$$M = \sum n_i$$

$$C = \sum i n_i$$

$$E(X) = \sum i \cdot \frac{n_i}{\sum n_i} \qquad\qquad E(Y) = \sum \frac{i^2 n_i}{\sum i n_i}$$

$$= \sum i \cdot \frac{n_i}{\sum n_i} \qquad\qquad = \sum i \cdot \frac{i n_i}{\sum i n_i}$$

Calculate $\dfrac{\dfrac{i n_i}{\sum i n_i}}{\dfrac{n_i}{\sum n_i}} = \dfrac{\dfrac{j n_j}{\sum i n_i}}{\dfrac{n_j}{\sum n_i}} = \dfrac{j n_j \left(\sum n_i\right)}{n_j \left(\sum i n_i\right)}$

$$= j \frac{\sum n_i}{\sum i n_i} = j \frac{M}{C}$$

$M \leq C$

$j \geq 1$

2. Discussion of how to finish moved to campuswire...