Theory of Probability

Oct 12, 2020

Discrete random variable X,

$P[X = x_i] = p(x_i)$ $\qquad \sum_{i=1}^{\infty} p(x_i) = 1$

Expected value of X:

$$E[X] = \sum_{i=1}^{\infty} x_i \, p(x_i)$$

weighted average of possible values.

If X is a R.V., then so is $Y = g(X)$.

$$E[Y] = \sum_{i=1}^{\infty} y_i \, P[Y = y_i] \qquad \text{if } y_i = g(x_i)$$

$$= \sum_{i=1}^{\infty} g(x_i) \, P[X = x_i]$$

$$= \sum_{i=1}^{\infty} g(x_i) \, p(x_i).$$

$$= E[\underbrace{g(X)}_{Y}]$$

Ex: $P[X = -1] = 1/3$

$P[X = 0] = 1/3$

$P[X = 1] = 1/3$

Let $Y = X^2$

Possible values of Y are $0, 1$

$P[Y = 0] = P[X = 0] = 1/3$

$P[Y = 1] = P[X = -1 \text{ or } X = 1]$

$= 2/3$.

Variance

- Expected value classifies a RV according to its average value.

$$Var[X] = E[(X - E[X])^2]$$
$$= E[X^2] - (E[X])^2.$$

$$Var[aX+b] = a^2 Var[X].$$

$$\underbrace{E[aX+b] = aE[X]+b}_{\text{is a linear transformation}}$$

The n^{th} moment of X is

$$E[X^n] = n^{th} \text{ moment.}$$

$\underline{n=1}$ $E[X^n] = E[X] = $ expected value

$\underline{n=2}$ $E[X^2] = Var[X] + (E[X])^2$

"Centered moments" $E[(\underbrace{X-\mu}_{X - E[X]})^n]$

<u>Example: 4.5 from Theoretical Exercises</u>

Let N be a nonnegative integer-valued random variable.

$$\left[P[N=j] = p(j) \quad , \quad j \geq 0 \right]$$

$$\sum_{j=1}^{\infty} p(j) = 1$$

(a) For nonnegative a_j, $j \geq 1$, show:

$$\sum_{j=1}^{\infty} (a_1 + \dots + a_j) P[N=j] = \sum_{i=1}^{\infty} a_i P[N \geq i].$$

$$E[N] = \sum_{j=0}^{\infty} j P[N=j]$$

$$= \sum_{j=1}^{\infty} j P[N=j].$$

Define the function $g(j) = \sum_{i=1}^{j} a_i$

$$E[g(N)] = \sum_{j=1}^{\infty} g(j) P[N=j]$$

$$= a_1 P[N=1]$$

$$+ (a_1 + a_2) P[N=2]$$

$$+ (a_1 + a_2 + a_3) P[N=3]$$

$$+ \dots$$

$$= a_1 \sum_{i=1}^{\infty} P[N=i]$$

$$+ a_2 \sum_{i=2}^{\infty} P[N=i]$$

$$+ a_3 \sum_{i=3}^{\infty} P[N=i]$$

$$= \sum_{j=1}^{\infty} a_j \sum_{i=j}^{\infty} P[N=i]$$

$$= \sum_{j=1}^{\infty} a_j P[N \geq j]$$

Show that $E[N] = \sum_{i=1}^{\infty} P[N \geq i]$

Set $a_i = 1$ for all i.

Then $E[g(N)] = \sum_{j=1}^{\infty} a_j P[N \geq j]$

$$= \sum_{j=1}^{\infty} P[N \geq j]$$

and $g(N) = \sum_{i=1}^{N} a_i = \sum_{i=1}^{N} 1 = N.$

$\Rightarrow \quad E[N] = \sum_{j=1}^{\infty} P[N \geq j].$

And finally, show that: $E[N(N+1)] = 2 \sum_{i=1}^{\infty} i \, P[N \geq i]$

So first note that

$$\sum_{i=1}^{N} i = 1 + 2 + 3 + 4 + \ldots + N-1 + N$$

N+1

N+1

$$= \frac{N(N+1)}{2}$$

Show that $E\left[\frac{N(N+1)}{2}\right] = \sum_{i=1}^{\infty} i \, P[N \geq i].$

Set $g(N) = \sum_{i=1}^{N} i = \frac{N(N+1)}{2}.$

$$\Rightarrow \quad E[g(N)] = E\left[\frac{N(N+1)}{2}\right] = \sum_{i=1}^{\infty} i \, P[N \geq i]$$