

Theory of Probability

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Discrete random variable X ,

$$P[X = x_i] = p(x_i)$$

$$\sum_{i=1}^{\infty} p(x_i) = 1$$

Expected value of X :

$$E[X] = \sum_{i=1}^{\infty} x_i P(x_i)$$

weighted average of possible values.

If X is a R.V., then so is $Y = g(X)$.

$$E[Y] = \sum_{i=1}^{\infty} y_i P[Y = y_i] \quad \text{if } y_i = g(x_i)$$

$$= \sum_{i=1}^{\infty} g(x_i) P[X = x_i]$$

$$= \sum_{i=1}^{\infty} g(x_i) p(x_i)$$

$$= E[\underbrace{g(X)}_Y]$$

Ex: $P[X = -1] = \frac{1}{3}$

$$P[X = 0] = \frac{1}{3}$$

$$P[X = 1] = \frac{1}{3}$$

Let $Y = X^2$

Possible values of Y

are 0, 1

$$P[Y = 0] = P[X = 0] = \frac{1}{3}$$

$$P[Y = 1] = P[X = -1 \text{ or } X = 1]$$

$$= \frac{2}{3}$$

[1]

Variance

- Expected value classifies a RV according to its average value.

$$\begin{aligned}\text{Var}[X] &= E[(X - E[X])^2] \\ &= E[X^2] - (E[X])^2.\end{aligned}$$

$$\text{Var}[aX + b] = a^2 \text{Var}[X].$$

$$E[aX + b] = aE[X] + b$$

is a linear transformation

The n^{th} moment of X is

$$E[X^n] = n^{\text{th}} \text{ moment.}$$

$$\underline{n=1} \quad E[X^n] = E[X] = \text{expected value}$$

$$\underline{n=2} \quad E[X^2] = \text{Var}[X] + (E[X])^2$$

"Centered moments" $E[\underbrace{(X - \mu)^n}_{X - E[X]}]$

Example: 4.5 from Theoretical Exercises

Let N be a nonnegative integer-valued random variable.

$$\left. \begin{aligned}P[N = j] &= p(j), \quad j \geq 0 \\ \sum_{j=0}^{\infty} p(j) &= 1\end{aligned} \right\}$$

(a) For nonnegative a_j , $j \geq 1$, show:

$$\sum_{j=1}^{\infty} (a_1 + \dots + a_j) P[N=j] = \sum_{i=1}^{\infty} a_i P[N \geq i].$$

$$\begin{aligned} E[N] &= \sum_{j=0}^{\infty} j P[N=j] \\ &= \sum_{j=1}^{\infty} j P[N=j]. \end{aligned}$$

Define the function $g(j) = \sum_{i=1}^j a_i$

$$\begin{aligned} E[g(N)] &= \sum_{j=1}^{\infty} g(j) P[N=j] \\ &= a_1 P[N=1] \\ &\quad + (a_1 + a_2) P[N=2] \\ &\quad + (a_1 + a_2 + a_3) P[N=3] \\ &\quad + \dots \\ &= a_1 \sum_{i=1}^{\infty} P[N=i] \\ &\quad + a_2 \sum_{i=2}^{\infty} P[N=i] \\ &\quad + a_3 \sum_{i=3}^{\infty} P[N=i] \\ &= \sum_{j=1}^{\infty} a_j \sum_{i=j}^{\infty} P[N=i] \\ &= \sum_{j=1}^{\infty} a_j P[N \geq j] \end{aligned}$$

Show that $E[N] = \sum_{i=1}^{\infty} P[N \geq i]$

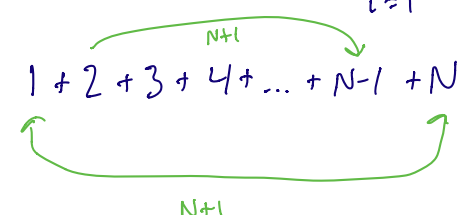
Set $a_i = 1$ for all i .

Then $E[g(N)] = \sum_{j=1}^{\infty} a_j P[N \geq j]$
 $= \sum_{j=1}^{\infty} P[N \geq j]$

and $g(N) = \sum_{i=1}^N a_i = \sum_{i=1}^N 1 = N$.

$\Rightarrow E[N] = \sum_{j=1}^{\infty} P[N \geq j]$.

And finally, show that: $E[N(N+1)] = 2 \sum_{i=1}^{\infty} i P[N \geq i]$

So first note that $\sum_{i=1}^N i = 1 + 2 + 3 + 4 + \dots + N-1 + N$

 $= \frac{N(N+1)}{2}$

Show that $E\left[\frac{N(N+1)}{2}\right] = \sum_{i=1}^{\infty} i P[N \geq i]$.

Set $g(N) = \sum_{i=1}^N i = \frac{N(N+1)}{2}$.

$\Rightarrow E[g(N)] = E\left[\frac{N(N+1)}{2}\right] = \sum_{i=1}^{\infty} i P[N \geq i]$