Theory of Probability — October 14, 2020

Bernoulli random variable X:

$$P[X=1] = p$$
$$P[X=0] = 1-p.$$

$$E[X] = p$$
$$Var[X] = E[(X-p)^2] = E[X^2 - 2Xp + p^2]$$
$$= p - 2p^2 + p^2$$
$$= p - p^2 = p(1-p).$$

If X_i is a Bernoulli random variable, then

$X = X_1 + X_2 + \dots + X_n$ is a binomial (n,p) random variable

$$P[X=k] = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[X] = np.$$
$$Var[X] = np(1-p).$$

Theoretical Exercise 4.15

A family has n children with probability αp^n ≥ 1,
with $0 \leq \alpha \leq (1-p)/p$.

$$P[C=k] = \alpha p^k.$$

(a) What proportion of families have no children?

$$P[C=0] = 1 - P[C \geq 1]$$
$$= 1 - \sum_{k=1}^{\infty} \alpha p^k.$$

$$= 1 - \underbrace{\sum_{k=1}^{\infty} \alpha p^k}.$$

$$S = p + \cancel{p^2} + \cancel{p^3} + \cancel{\ldots}$$

$$pS = \cancel{p^2} + \cancel{p^3} + \cancel{p^4} + \ldots$$

$$(S - pS) = p$$

$$S(1-p) = p \quad \Rightarrow$$

$$S_n = \sum_{1}^{n} p^k$$

$$pS_n = \sum_{1}^{n} p^{k+1}$$

$$S_n - pS_n = p - p^{n+1}$$

$$\Rightarrow S_n(1-p) = p - p^{n+1}$$

$$S_n = \frac{p}{1-p} - p^{n+1}\frac{1}{1-p}$$

$$\lim_{n\to\infty} S_n = \frac{p}{1-p} \quad \text{since} \quad p^{n+1} \to 0. \quad \text{since } p < 1.$$

$$\Rightarrow \quad = 1 - \frac{\alpha p}{1-p}.$$

$$\Rightarrow P[C=0] = 1 - \frac{\alpha p}{1-p} = \frac{1 - p - \alpha p}{1-p} = \frac{1-(1+\alpha)p}{1-p}.$$

(b) If each child is equally likely to be a boy or a girl, what proportion of families consist of k boys?

$$\left.\begin{aligned} P[B=k] = {} & P[B=k \mid C=k]\, P[C=k] \\ & + P[B=k \mid C=k+1]\, P[C=k+1] \\ & + \ldots \end{aligned}\right\} \text{Law of Total Probability}$$

$$= \sum_{j=k}^{\infty} P[B=k \mid C=j]\, \alpha p^j$$

$$= \sum_{j=k}^{\infty} \left(\binom{j}{k} \left(\tfrac{1}{2}\right)^{k} \left(\tfrac{1}{2}\right)^{j-k} \right) \alpha p^{j}$$

$$= \sum_{j=k}^{\infty} \binom{j}{k} \left(\tfrac{1}{2}\right)^{j} \alpha p^{j}$$

$$= \sum_{j=k}^{\infty} \binom{j}{k} \left(\tfrac{p}{2}\right)^{j} \alpha$$

Self-test Exercise 13

7 Person panel of judges, each makes correct judgement independently with Probability 0.7.

$$P[J_i=1] = 0.7. \qquad P[J_i=0] = 0.3.$$

Decision by majority, what is the probability the panel makes the correct decision?

$$P[\underbrace{J_1+J_2+\dots+J_7}_{J} \geq 4] = ?$$

$\Rightarrow$ then $J \sim \text{binomial}(7, 0.7)$

$$P[J \geq 4] = P[J=4] + P[J=5] + P[J=6] + P[J=7].$$

$$= \sum_{k=4}^{7} \binom{7}{k} (0.7)^{k} (0.3)^{7-k}$$

$$= 0.87$$

Given that 4 judges agreed, what is the probability that the panel made the correct decision?

$$P[J=4 \mid J=4 \text{ or } J=3]$$

$$= \frac{P[J=4 \cap (J=4 \text{ or } J=3)]}{P[J=4 \text{ or } J=3]}$$

$$= \frac{P[J=4]}{P[J=4] + P[J=3]}$$

$$= \frac{\binom{7}{4}(0.7)^4(0.3)^3}{\binom{7}{4}(0.7)^4(0.3)^3 + \binom{7}{3}(0.7)^3(0.3)^4}$$

$$= \frac{1}{1 + \frac{\cancel{\binom{7}{3}}}{\cancel{\binom{7}{4}}}\frac{\cancel{(0.7)^3}}{(0.7)^{\cancel{4}\,1}}\frac{(0.3)^{\cancel{4}\,1}}{\cancel{(0.3)^3}}}$$

$$= \frac{1}{1 + \frac{.3}{.7}} = \frac{.7}{.7 + .3} = .7$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n!}{(n-k)!\,(n-(n-k))!}$$

$$= \frac{n!}{(n-k)!\,k!}$$

$$= \binom{n}{n-k}$$

$$P[J=5 \mid J=5 \text{ or } J=2]$$

$$= \frac{P[J=5]}{P[J=5] + P[J=2]}$$

$$= \frac{\binom{7}{5}\cancel{(0.7)^5}(0.3)^2}{\cancel{\binom{7}{5}(0.7)^5(0.3)^2} + \binom{7}{2}(0.7)^2(0.3)^5}$$

$$= \frac{1}{1 + \frac{\cancel{(0.7)^2}}{(0.7)^3} \frac{(0.3)^3}{\cancel{(0.3)^2}}}$$

$$= \frac{1}{1 + \left(\frac{0.3}{0.7}\right)^3} \approx .93$$