Theory of Probability

Oct 28, 2020

Negative binomial : (r, p)

successes

probability of success

$$P[X = n] = \binom{n-1}{r-1} p^r (1-p)^{n-r}$$

With regard to Theoretical exercise 4.22

Team A wins a game with probability p.

B $(1-p)$.

$P[$A wins 2 games first$] =$

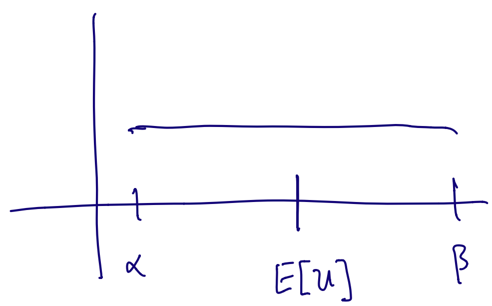
$U \sim \text{Uniform}(\alpha, \beta)$ if

$$P[U \in (a,b)] = \int_a^b \frac{1}{\beta-\alpha}\,dx$$

$\alpha < a < b < \beta$

PDF is a constant.

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha}, & x \in (\alpha,\beta) \\ 0, & \text{otherwise} \end{cases}$$



$$E[U] = \frac{1}{2}(\alpha+\beta)$$

$$Var[U] = E[U^2] - (E[U])^2$$

$$= \int_\alpha^\beta u^2 \frac{1}{\beta-\alpha}\,du - \frac{1}{4}(\alpha+\beta)^2$$

$$= \frac{1}{3}u^3 \frac{1}{\beta-\alpha}\Big|_{\alpha}^{\beta} - \frac{1}{4}(\alpha+\beta)^2$$

$$= \frac{1}{3(\cancel{\beta-\alpha})}\underbrace{(\beta^3-\alpha^3)}_{(\cancel{\beta-\alpha})(\beta^2+\alpha\beta+\alpha^2)} - \frac{1}{4}(\alpha+\beta)^2$$

$$= \frac{4(\beta^2+\alpha\beta+\alpha^2) - 3(\alpha^2+2\alpha\beta+\beta^2)}{12}$$

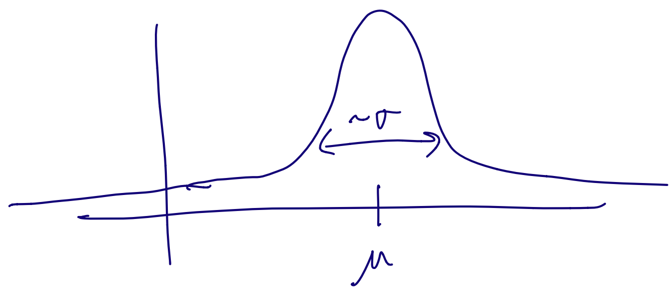
$$= \frac{\alpha^2 - 2\alpha\beta + \beta^2}{12} = \frac{(\alpha-\beta)^2}{12}$$

<u>Normal Random Variable</u>

PDF: $f(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$

$E[X] = \mu$

$Var[X] = \sigma^2$



If $X \sim N(\mu, \sigma^2)$, then

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

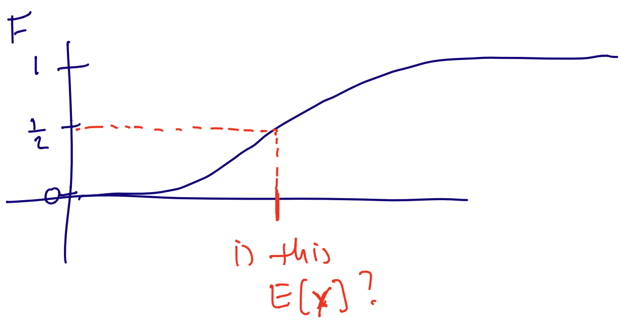
CDF for Z is $P[Z \le z] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = \Phi(z)$.

$$P[Z \in (a,b]] = P[Z \le b] - P[Z \le a]$$
$$= \Phi(b) - \Phi(a).$$

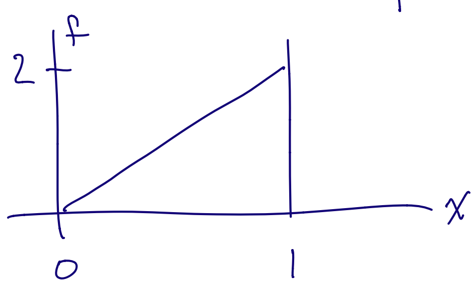
If $X \sim N(\mu, \sigma^2)$, then

$$P[X \le x] = P\left[\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right]$$
$$= P\left[Z \le \frac{x-\mu}{\sigma}\right] = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Question: Is $E[X] = x$ such that $\overbrace{P[X \le x]}^{F(x)} = \frac{1}{2}$.



This is generally only true if pdf is even about the point $x = E[X]$.



$$f(x) = 2x$$

$$F(x) = \int_0^x 2t\, dt = x^2 \Rightarrow F(x) = \frac{1}{2}$$
$$\Rightarrow x = \frac{1}{\sqrt{2}}$$

$$E[X] = \int_0^1 x \cdot 2x\, dx = \int_0^1 2x^2\, dx = \frac{2}{3}$$

If f is symmetric about μ, then $f(x-\mu)$ is symmetric about 0 $\Rightarrow \int x\, f(x-\mu)\, dx = 0 \Rightarrow E[X-\mu] = 0$
$\Rightarrow E[X] = \mu.$

Normal approximation to the binomial distribution:

If $S_n \sim \text{binomial}(n,p)$ then

$$\frac{S_n - np}{\sqrt{np(1-p)}} \rightsquigarrow N(0,1) \quad \text{as} \quad n \to \infty$$

$$\lim_{n\to\infty} P\left[a < \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right] = \Phi(b) - \Phi(a)$$

$$= P[a < Z \le b] \quad \text{with } Z \sim N(0,1).$$