Theory of Probability

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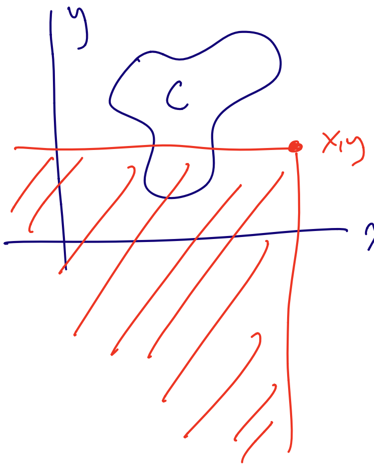
$$P[(X,Y)\in C] = \iint_C f(x,y)\,dx\,dy.$$



$$F(x,y) = P[X \le x,\ Y \le y]$$

$$= \int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)\,du\,dv$$

$f(x,y) \sim$ pdf of X,Y

$f_X \sim$ pdf of X

$$\boxed{f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy}$$

$$\frac{\partial^2 F}{\partial x\,\partial y} = f$$

Functions of a continuous random variable:

If f is the pdf of X, F the CDF, and $Y = g(X)$, then what is the pdf/CDF of Y?

$$P[Y \le y] = P[g(X) \le y] = P[X \le g^{-1}(y)]$$

$$= \underbrace{\int_{-\infty}^{g^{-1}(y)} f_X(x)\,dx}\ = \underset{\text{CDF of } Y}{F_Y(y)}$$

We know that

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Monotonic Increasing. If $y > x$, then $g(y) > g(x)$.

Monotonic non decreasing: If $y > x$, then $g(y) \geq g(x)$.

Fundamental theorem of calculus:

$$F(y) = \int_a^{h(y)} f(x)\, dx$$

What is F'? $\rightarrow f(h(y))\, h'(y)$

$$F_Y(y) = \int_{-\infty}^{g^{-1}(y)} f(x)\, dx$$

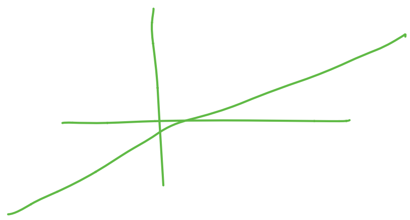
Let $x = g^{-1}(z)$ $\quad g(x) = z$

$x : -\infty \rightarrow g^{-1}(y)$

$z : -\infty \rightarrow y$

(Assume g is monotonic increasing and differentiable)

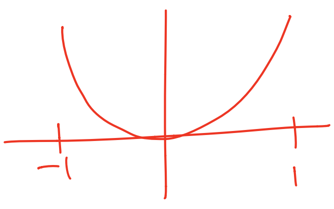
$$= \int_{-\infty}^{y} \underbrace{f(g^{-1}(z))\, \frac{dg^{-1}}{dz}(z)}\, dz$$



is the pdf of Y.

$X \sim \text{Uniform}(-1,1) \Rightarrow f_x(x) = \begin{cases} \frac{1}{2} & \text{on } [-1,1) \\ 0 & \text{otherwise} \end{cases}$

$Y = X^2$



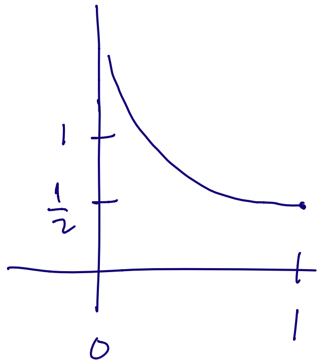
By previous calculation:

$$P[Y \leq y] = P[X^2 \leq y] = \cancel{P[X \leq \sqrt{y}]}$$

$$= P[-\sqrt{y} \leq X \leq \sqrt{y}] \qquad y \geq 0$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f_x(x)\, dx.$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}\,dx = \frac{x}{2}\Big|_{-\sqrt{y}}^{\sqrt{y}} = \sqrt{y}$$



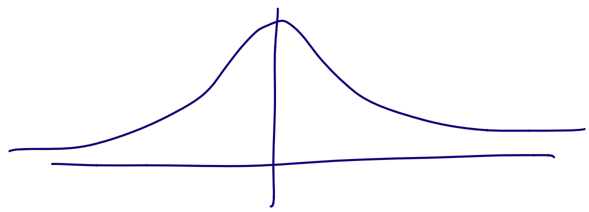
$$f_Y(y) = \frac{d}{dy}\left(\sqrt{y}\right) = \frac{1}{2}\frac{1}{\sqrt{y}}$$

$$\int_0^1 f_Y(y)\,dy = \int_0^1 \frac{1}{2}\frac{1}{\sqrt{y}}\,dy$$

$y^{-1/2}$

$$= \frac{1}{\cancel{2}}\,\cancel{2}\,y^{1/2}\Big|_0^1 = 1$$

Let $X \sim N(0,1)$



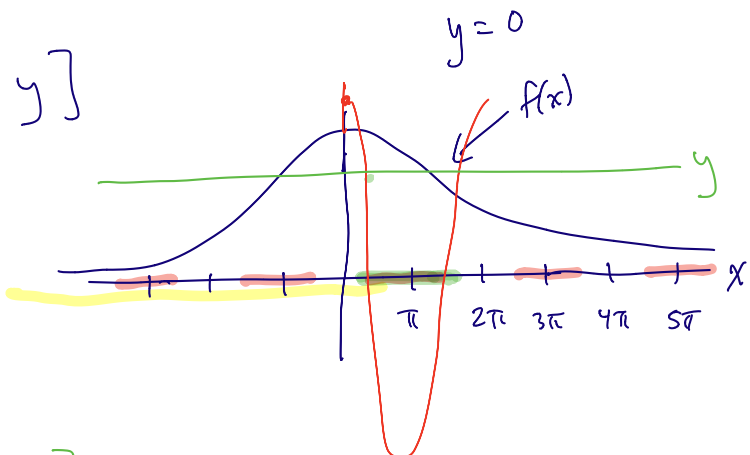
$Y = \cos X$

$= g(X)$

$X \in (-\infty, \infty)$

$Y \in [-1, 1]$

$$P[Y \le y] = P[\cos X \le y]$$



$$= \int_{\cos x \le y} f(x)\,dx$$

$$\ne P[X \le a\cos y]$$

$$\ne \int_{-\infty}^{a\cos y} f(x)\,dx$$