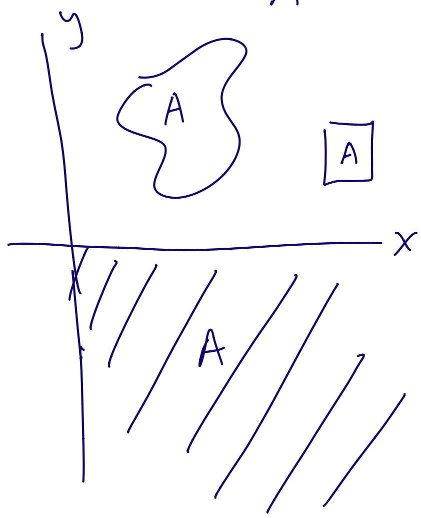
Theory of Probability

Nov 9, 2020

Joint continuous distributions:

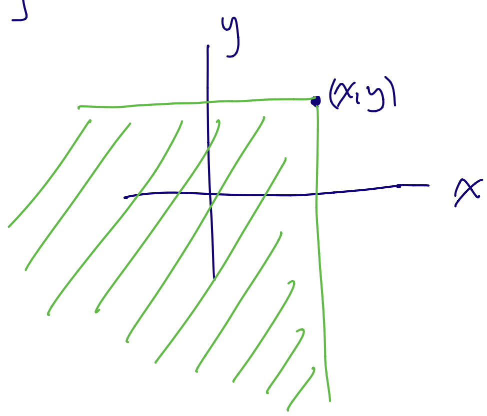
$$P[(X,Y) \in A] = \iint_A \underbrace{f(x,y)}_{\text{joint probability density function}} \, dx \, dy$$



The distribution function is defined similarly:

$$F(x,y) = P[X \le x, Y \le y]$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} f(u,v) \, du \, dv$$



$$\Rightarrow \quad f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y).$$

$$\underbrace{f_X(x)}_{\text{density for } X} = \int_{-\infty}^{\infty} f(x,y) \, dy \qquad f_Y(x) = \int_{-\infty}^{\infty} f(x,y) \, dx.$$

Independence of Random Variables.

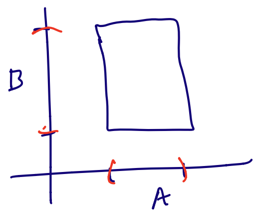
Recall: Events A, B were independent if

$$P[A|B] = P[A] \iff P[AB] = P[A]\,P[B].$$

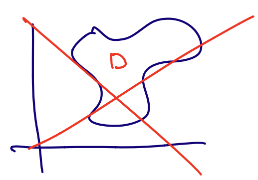
Two random variables X, Y are independent if

$$P[X \in A,\, Y \in B] = P[X \in A]\, P[Y \in B].$$

$$= \int_A \int_B f(x,y)\, dy\, dx \Rightarrow$$



$$= \int_A \left(\int_B f(x,y)\, dy \right) dx$$



If X, Y are independent:

$$P[X, Y \in D] = \iint_{D(x,y)} f_X(x)\, f_Y(y)\, dx\, dy$$

① $f(x,y) = f_X(x)\, f_Y(y)$

② $F(x,y) = F_X(x)\, F_Y(y)$.

Sums of Independent Random Variables

Let X, Y be independent random variables.

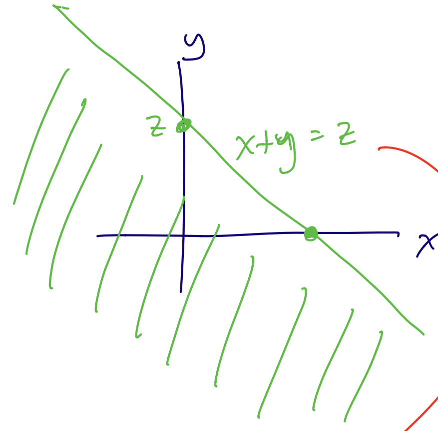
$$\Rightarrow f(x,y) = f_X(x)\, f_Y(y).$$

Consider $Z = X + Y$

What is $F_Z(z) = P[Z \le z]$?

$$P[Z \le z] = P[X+Y \le z]$$

$$= \iint_{x+y \le z} f(x,y)\,dx\,dy$$



$y = z - x$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} \underbrace{f_X(x)}_{\text{does not depend on } y}\, f_Y(y)\,dx\,dy$$

$$P[Z \le z] = \int_{-\infty}^{\infty} F_X(z-y)\, f_Y(y)\,dy = F_Z(z)$$

We know that $f_Z(z) = \frac{d}{dz} F_Z(z)$

$$= \frac{d}{dz}\int_{-\infty}^{\infty} F_X(z-y)\, f_Y(y)\,dy$$

$$= \underbrace{\int_{-\infty}^{\infty} f_X(z-y)\, f_Y(y)\,dy}_{\text{convolution of } f_X \text{ and } f_Y.}$$

If instead we wanted the density for $\underbrace{W+X+Y}_{U}$

$\Rightarrow$

$$P[U \le u] = P[W+Z \le u]$$

$$= \int F_w(u-z)\, f_z(z)\, dz$$

$$\Rightarrow \quad f_u = \int f_w(u-z)\, f_z(z)\, dz$$

$$= \underbrace{\int_{-\infty}^{\infty} f_w(u-z) \int_{-\infty}^{\infty} f_X(z-y)\, f_Y(y)\, dy\, dz}_{\text{iterated convolution}}$$

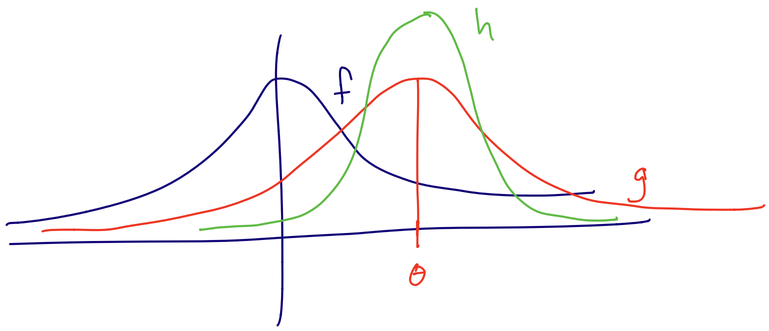
Ex:

$$\text{Gamma}(s,\lambda) + \text{Gamma}(t,\lambda) \sim \text{Gamma}(s+t,\lambda)$$

$$N(\mu_1,\sigma_1^2) + N(\mu_2,\sigma_2^2) \sim N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2)$$

Cauchy

$$f(x) = \frac{1}{\pi}\,\frac{1}{1+x^2}$$



$$g(x) = f(x-\theta)$$

$$= \frac{1}{\pi}\,\frac{1}{1+(x-\theta)^2}$$

$$h(x) = f\left(\frac{x-\theta}{\tau}\right)$$

$$= \frac{1}{\pi}\,\frac{1}{1+\frac{(x-\theta)^2}{\tau^2}} = \frac{\tau^2}{\pi}\,\frac{1}{\tau^2+(x-\theta)^2}$$