Theory of Probability

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Conditional Distributions

Recall:

$$P[E|F] = \frac{P[E \cap F]}{P[F]}$$

analogous definition

Discrete Case:

If $P[X=x, Y=y] = p(x,y)$ probability mass function

$$P[X=x|Y=y] = \frac{P[X=x, Y=y]}{P[Y=y]} = \frac{p(x,y)}{p_Y(y)}$$

$$= p_{X|Y}(x|y)$$

Continuous Case:

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

Independence : $f(x,y) = f_X(x)\, f_Y(y)$

$$\Rightarrow \quad f_{X|Y}(x|y) = \frac{f_X(x)\, \cancel{f_Y(y)}}{\cancel{f_Y(y)}} = f_X(x).$$

$$P[X = x] = 0$$

$$P[X \in (x, x+dx)] = \int_x^{x+dx} f(u)\, du$$

$$\approx f(x)\, dx$$

$$f(x,y) = \begin{cases} \dfrac{e^{-x/y}\, e^{-y}}{y} & x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

What is $P[X > 1 \mid Y = y]$?

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_0^\infty f(x,y)\, dx$$

$$= \int_0^\infty \frac{e^{-x/y}\, e^{-y}}{y}\, dx$$

$$= \frac{e^{-y}}{y} \int_0^\infty e^{-x/y}\, dx$$

$$= \frac{e^{-y}}{y} \left[-y e^{-x/y} \right]_0^\infty = \frac{e^{-y}}{y} \left(0 + y e^{0} \right) = e^{-y}$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{e^{-x/y}\,\cancel{e^{-y}}/y}{\cancel{e^{-y}}}$$

$$= \frac{e^{-x/y}}{y}$$

$$P[X>1 \mid Y=y] = \int_1^\infty f_{X|Y}(x|y)\,dx$$

$$= \int_1^\infty \frac{e^{-x/y}}{y}\,dx$$

$$= \frac{1}{y}\Big[-y e^{-x/y}\Big]_1^\infty$$

$$= \frac{1}{y}\left[0 + y e^{-1/y}\right]$$

$$= \boxed{e^{-1/y}}$$

<u>Ex 5c from Text : t-distribution</u>

If Z, Y are independent, and

$Z \sim N(0,1)$, $Y \sim \chi^2_n$

If $Z_1, Z_2, \dots, Z_n$ are iid $N(0,1)$, then

$Y = Z_1^2 + Z_2^2 + Z_3^2 + \dots + Z_n^2 \sim \chi^2_n$

Then $\underline{T} = \dfrac{Z}{\sqrt{Y/n}} = \sqrt{\dfrac{n}{Y}}\, Z$ is

said to be a "t-distribution with n deg. of freedom.".

How do we compute its density? (f_T)

What do we know?

- f_Z
- f_Y
- Z, Y independent

The marginal f_T can be written as

$$f_T(t) = \int_0^\infty f_{T,Y}(t,y)\, dy$$

① Compute the distribution of $T|Y$ to obtain the joint distribution of T, Y

$$f_{T,Y}(t,y) = f_{T|Y}(t|y)\, f_Y(y).$$

② Integrate $f_{T,Y}$ in y to obtain f_T.

① With $Y=y$, $T = \underbrace{\sqrt{\frac{n}{y}}}_{\text{a number}} Z \sim N(0, \frac{n}{y})$

Z: $N(0,1)$

$$\Rightarrow T \mid Y=y \sim N(0, \tfrac{n}{y})$$

$$\Rightarrow f_{T|Y}(t|y) = \frac{1}{\sqrt{2\pi n/y}} e^{-t^2 y/2n} \quad \checkmark$$

From Ex 3b in Ross:

$$f_Y(y) = \frac{e^{-y/2}\, y^{n/2-1}}{2^{n/2}\, \Gamma(n/2)} \qquad (\text{Gamma density}).$$

So
$$f_{T,Y}(t,y) = f_{T|Y}(t|y)\, f_Y(y)$$

$$= \frac{1}{\sqrt{2\pi n}\; 2^{n/2}\, \Gamma(n/2)} \underbrace{e^{-t^2 y/2n}\, e^{-y/2}}\, y^{(n-1)/2}$$

$$= \frac{1}{\sqrt{2\pi n}\; 2^{n/2}\, \Gamma(n/2)}\, e^{-\frac{t^2+n}{2n} y}\, y^{(n-1)/2}$$

Let $c = \frac{t^2+n}{2n}$ and integrate in y:

$$f_T(t) = \int_0^\infty \frac{1}{\sqrt{2\pi n}\; 2^{n/2}\, \Gamma(n/2)}\, e^{-cy}\, y^{(n-1)/2}\, dy$$

Let $u = cy$
$y = \frac{1}{c} u$
$dy = \frac{du}{c}$

$$= \frac{1}{\sqrt{2\pi n}\; 2^{n/2}\, \Gamma(n/2)} \int_0^\infty e^{-u} \frac{1}{c^{(n-1)/2}}\, u^{(n-1)/2}\, \frac{du}{c}$$

$$= \frac{1}{\sqrt{2\pi n}\; 2^{n/2}\, \Gamma(n/2)} \frac{1}{c^{(n+1)/2}} \int_0^\infty e^{-u}\, u^{(n-1)/2}\, du$$

$$= \boxed{\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\; \Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}} \qquad t \in (-\infty, \infty).$$