Theory of Probability

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Order statistics

has density $f(x)$.

continuous

Let $X_1, X_2, \ldots, X_n$ be IID random variables

Let $X_{(1)}, X_{(2)}, X_{(3)}, \ldots$
$\Longleftrightarrow X_{(1)} \le X_{(2)} \le X_{(3)} \le \ldots$ be the order statistics of $X_1, \ldots, X_n$.

$$X_{(1)} = \min(X_1, \ldots, X_n)$$

$$X_{(2)} = \text{next smallest}$$

$$\vdots$$

$$\Rightarrow \quad f_{X_{(1)} X_{(2)} \cdots X_{(n)}}(x_1, \ldots, x_n) = n!\, f(x_1) f(x_2) \cdots f(x_n)$$

on the set $x_1 < x_2 < x_3 < \ldots < x_n$

Ex: $X_1, X_2, X_3 \sim U(0,1)$

$X_{(1)} < X_{(2)} < X_{(3)}$ be the order statistics

$f(x) = 1$ on $x \in (0,1)$

$$\Rightarrow \quad f_{X_{(1)} X_{(2)} X_{(3)}}(x_1, x_2, x_3) = 3! \quad \text{on the set}$$

$$0 < x_1 < x_2 < x_3 < 1.$$

Check that this is indeed a prob. density.

$$\underset{0<x_1<x_2<x_3<1}{\iiint} 3!\, dx_1\, dx_2\, dx_3 = \int_0^1 \int_0^{x_3} \int_0^{x_2} 3!\, dx_1\, dx_2\, dx_3$$

$$= \int_0^1 \int_0^{x_3} 6x_1 \Big|_0^{x_2} dx_2\, dx_3$$

$$= \int_0^1 \int_0^{x_3} 6x_2\, dx_2\, dx_3$$

$$= \int_0^1 3x_2^2 \Big|_0^{x_3} dx_3$$

$$= \int_0^1 3x_3^2\, dx_3$$

$$= x_3^3 \Big|_0^1 = 1$$

Ex: $$E[X_{(1)}] = \int_{-\infty}^{\infty} \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} x_1\, n!\, f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_n$$

Functions of Several Random Variables

Change of variables in multiple integrals:

$\iint f(x,y)\, dx\, dy$ change to polar coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$dx\,dy = J\, dr\, d\theta$$

$$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\, d\theta$$

$$dx\,dy = \left(\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}\right) dr\,d\theta$$

$$= \left(\cos\theta \; r\cos\theta + r\sin\theta\,\sin\theta\right) dr\,d\theta$$

$$= \left(r\cos^2\theta + r\sin^2\theta\right) dr\,d\theta$$

$$= r\,dr\,d\theta$$

$$\iint f(x,y)\,dx\,dy = \iint f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta .$$

If X_1, X_2 are continuous random variables with joint pdf $f(x_1, x_2)$, and if

$$Y_1 = g_1(X_1, X_2)$$
$$Y_2 = g_2(X_1, X_2)$$

the mapping

$$\begin{matrix} x_1 \\ x_2 \end{matrix} \longrightarrow \begin{matrix} g_1(x_1, x_2) = y_1 \\ g_2(x_1, x_2) = y_2 \end{matrix}$$

is cont. diff. and invertible.

then what is the joint pdf of Y_1, Y_2?

Start with the distribution function:

$$P[Y_1 \le y_1, Y_2 \le y_2] = P[g_1(X_1, X_2) \le y_1, g_2(X_1, X_2) \le y_2]$$

$$= \iint f(x_1, x_2)\, dx_1\, dx_2$$

defines "some" region of integration
$$\begin{cases} g_1(x_1,x_2) \le y_1 & \Rightarrow u \le y_1 \\ g_2(x_1,x_2) \le y_2 & \Rightarrow v \le y_2 \end{cases}$$

Change Variables:

Set $\begin{cases} u = g_1(x_1, x_2) \\ v = g_2(x_1, x_2) \end{cases}$

$$\Rightarrow du\,dv = \underbrace{\begin{vmatrix} \dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_1}{\partial x_2} \\ \dfrac{\partial g_2}{\partial x_1} & \dfrac{\partial g_2}{\partial x_2} \end{vmatrix}}_{J} dx_1\, dx_2$$

$$\Rightarrow x_1 = h_1(u,v)$$
$$x_2 = h_2(u,v)$$

$$\Rightarrow \quad dx_1\, dx_2 = \frac{1}{J}\, du\, dv$$

$$\Rightarrow \int_{-\infty}^{y_2}\int_{-\infty}^{y_1} f\big(h_1(u,v),\, h_2(u,v)\big)\, \frac{1}{J}\, du\, dv \qquad = F_{Y_1 Y_2}(y_1, y_2)$$

$$\Rightarrow f_{Y_1 Y_2}(y_1, y_2) = \frac{\partial^2 F_{Y_1 Y_2}}{\partial y_1\, \partial y_2}$$

$$= f\big(h_1(y_1,y_2),\, h_2(y_1,y_2)\big)\, \frac{1}{J}$$