Theory of Probability Nov 23, 2020

For a collection of random variables $X_1, .., X_n$,

If $Y = X_1 + X_2 + .. + X_n$, then

$$E[Y] = E[X_1 + .. + X_n]$$
$$= \sum_{i=1}^{n} E[X_i]$$

Sample Mean: If X_i are IID R.V's, each with $E[X_i] = \mu$,

$$\Rightarrow \quad E\left[\frac{1}{n}(X_1 + ... + X_n)\right] = \frac{1}{n}\left(\sum_{i=1}^{n} E[X_i]\right)$$

$$\boxed{\bar{X} = \frac{1}{n}\sum_{1}^{n} X_i} \qquad = \frac{1}{n} n \cdot \mu = \mu.$$

$$E[X+Y] = \iint (x+y)\, f(x,y)\, dx\, dy$$
$$= \iint x\, f(x,y)\, dx\, dy + \iint y\, f(x,y)\, dx\, dy$$
$$= \underbrace{\int x\, f_X(x)\, dx}_{E[X]} + \underbrace{\int y\, f_Y(y)\, dy}_{E[Y]}.$$

Covariance

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

If X,Y are independent then

$$Cov(X,Y) = E[X-\mu_X] \cdot E[Y-\mu_Y]$$

$$= 0 \cdot 0$$

Correlation

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \in [-1,1].$$

$$Cov(aX,Y) = a\,Cov(X,Y)$$

$$\Rightarrow \rho(aX,Y) = \frac{Cov(aX,Y)}{\sqrt{Var(aX)}\sqrt{Var(Y)}}$$

$$= \frac{\cancel{a}\,Cov(X,Y)}{\sqrt{\cancel{a^2}\,Var(X)}\sqrt{Var(Y)}} = \frac{a}{|a|}\rho(X,Y).$$

If $a<0$, then $Var(aX) = a^2 Var(X)$.

and $stdev(aX) = \sqrt{a^2 Var(X)} > 0$

$$= |a|\sqrt{Var(X)}$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

$$= Cov\left(\frac{X}{\sqrt{Var(X)}}, \frac{Y}{\sqrt{Var(Y)}}\right)$$

$$Var\left(\frac{X}{\sqrt{Var(X)}}\right) = 1$$

Sample Variance

If $X_1, \dots, X_n$ are IID R.V.s with mean μ and variance σ^2, then the sample variance is given by:

$$S^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}$$

$$E[S^2] = \sigma^2$$

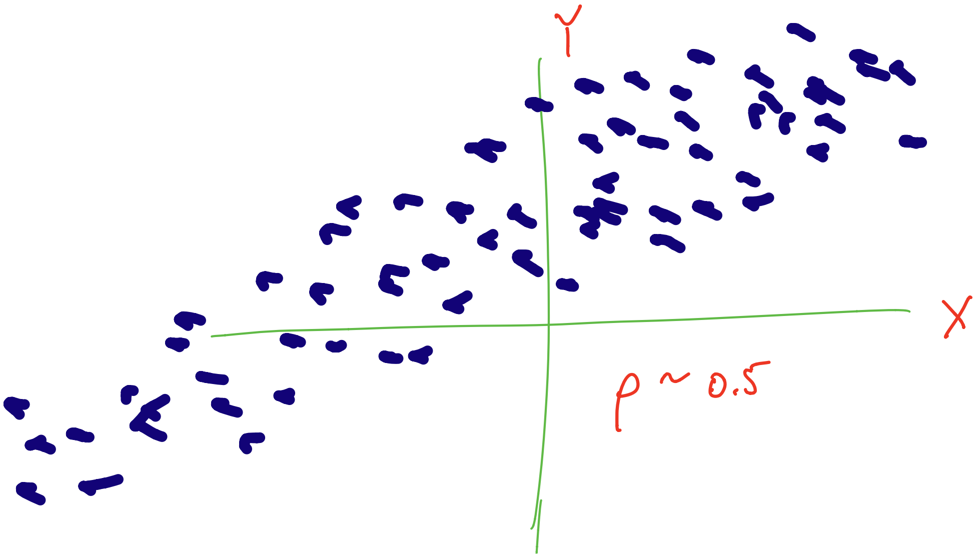
$$E[\bar{X}] = \mu$$

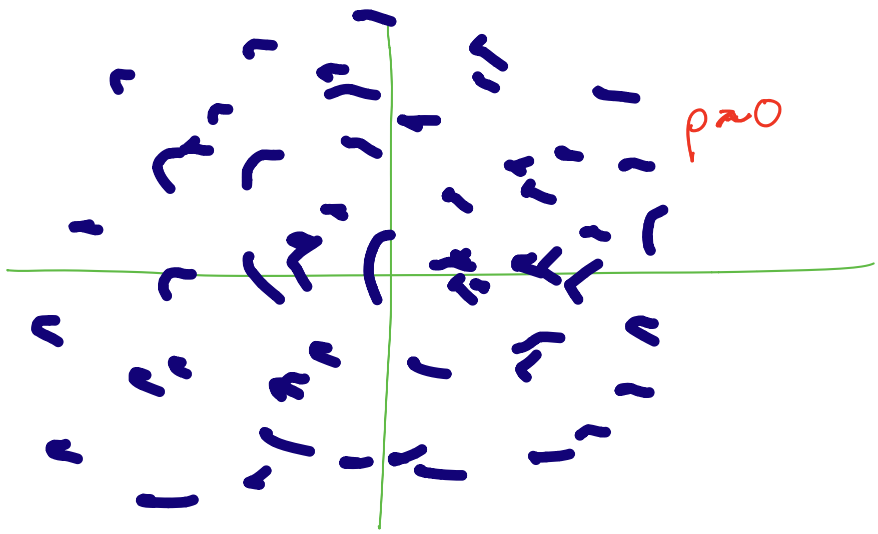
Ex:

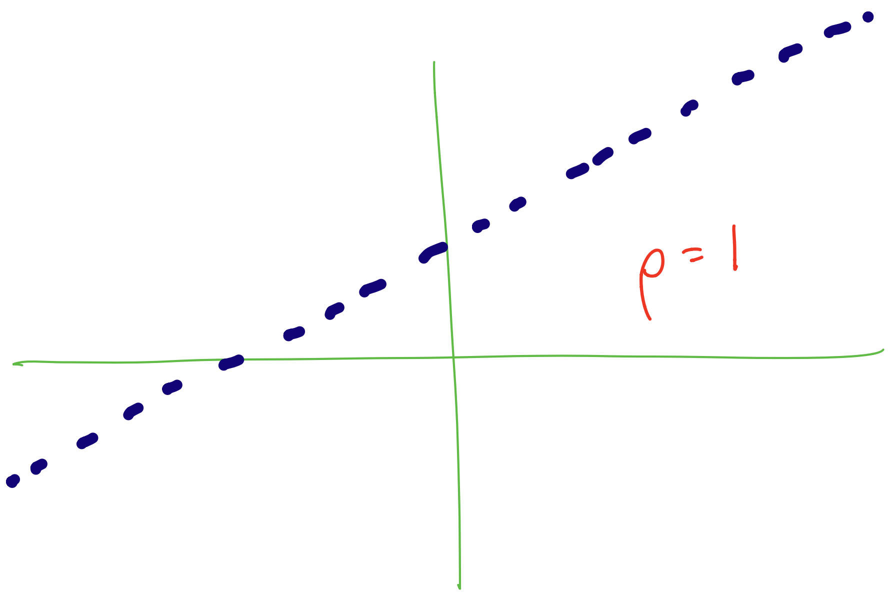
$$Var(\bar{X}) = Var\left(\frac{1}{n}(X_1 + \dots + X_n)\right) = \frac{1}{n^2} Var\left(\sum_1^n X_i\right)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2} \sum_1^n \sigma^2 = \frac{n\sigma^2}{n^2}$$

$$= \boxed{\frac{\sigma^2}{n}}$$


Y
X
$\rho \sim 0.5$


$\rho \approx 0$


$\rho = 1$

Theo Exercise 7.23:

If $Y = a + bX$, what is $\rho(X,Y)$?

$Var(X) = \sigma^2$ $\qquad E[X] = \mu$.

$Var(Y) = Var(a + bX)$

$= b^2\sigma^2$

$Cov(X,Y) = Cov(X, a + bX)$

$= E[(X-\mu)(\cancel{a}+bX - (\cancel{a}+b\mu))]$

$= E[(X-\mu)(bX - b\mu)]$

$= b \cdot E[(X-\mu)(X-\mu)]$

$= b \cdot E[(X-\mu)^2]$

$= b \cdot \sigma^2$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{var(X)}\sqrt{var(Y)}} = \frac{b\sigma^2}{\sigma \cdot |b|\sigma} = \frac{b}{|b|}$$

$$= \begin{cases} 1 & \text{if } b > 0 \\ -1 & \text{if } b < 0 \end{cases}$$

Theo Exercise 7.4

Let X be a r.v. with

$$E[X] = \mu < \infty$$

$$Var[X] = \sigma^2 < \infty$$

$g = g(x)$ is twice differentiable

Approximate $E[g(X)]$.

Near μ, $g(x) \approx g(\mu) + g'(\mu)(x-\mu) + \frac{g''(\mu)}{2}(x-\mu)^2$.

$$E[g(X)] \simeq E\left[g(\mu) + g'(\mu)(X-\mu) + \frac{g''(\mu)}{2}(X-\mu)^2\right]$$

$$= g(\mu) + \cancel{g'(\mu)E[X-\mu]} + \frac{g''(\mu)}{2}E[(X-\mu)^2]$$

$$= \boxed{g(\mu) + \frac{g''(\mu)}{2}\sigma^2}$$