Theory of Probability Nov 30, 2020

Moment generating Function:

For a random variable X,

$$M_X(t) = E[e^{tX}]$$

indicator function

$$= \int_a^b e^{tx} f(x)\, dx = \int_{-\infty}^{\infty} e^{tx} f(x)\, \mathbb{1}_D(x)\, dx$$

Compare with the Laplace Transform:

$$\mathcal{L}f(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$\Rightarrow \quad M(0) = \int e^0 f(x)\, dx = 1$$

$$M'(t) = \int x e^{tx} f(x)\, dx$$

$$M'(0) = \int x f(x)\, dx = E[X].$$

$$M^{(n)}(0) = \int x^n f(x)\, dx$$

$$= E[X^n].$$

$$\varphi_X(t) = E[e^{itX}]$$

$$= \int e^{itx} f(x)\, dx \qquad \text{Fourier transfor of } f$$

Characteristic function.

If X, Y are independent, then set $Z = X+Y$

$$\Rightarrow M_Z(t) = E[e^{tZ}]$$
$$= E[e^{t(X+Y)}]$$
$$= E[e^{tX} e^{tY}]$$
$$= \iint e^{tx} e^{ty} \underline{f(x,y)}\, dx\, dy$$
$$= \iint e^{tx} e^{ty} f_X(x) f_Y(y)\, dx\, dy$$
$$= \underline{M_X(t)}\ \underline{M_Y(t)}.$$

Multivariate Normal Random Variables

$Z_1, \dots, Z_n$ are $N(0,1)$ and independent:

$$\vec{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix} = A\vec{Z} + \vec{\mu} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \\ a_{m1} & & & a_{mn} \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}.$$

$$X_j \sim N\left(\mu_j, \sum_{k=1}^{n} a_{jk}^2\right)$$

Normal random variables are invariant under affine transformations.

Since the Z_js were independent, they are also uncorrelated: $\mathrm{Cov}(Z_j, Z_k) = 0$ if $j \neq k$.

$$\Rightarrow \quad \mathrm{Cov}(X_i, X_j) = \mathrm{Cov}\Big(\mu_i + \sum_k a_{ik} Z_k,\ \mu_j + \sum_{k'} a_{jk'} Z_{k'}\Big)$$

$$= E\Big[\Big(\sum_k a_{ik} Z_k\Big)\Big(\sum_{k'} a_{jk'} Z_{k'}\Big)\Big]$$

$$= E\Big[\sum_{k,k'} a_{ik} a_{jk'} Z_k Z_{k'}\Big]$$

$$= \sum_{k,k'} a_{ik} a_{jk'} E[Z_k Z_{k'}]$$

$$= \sum_k a_{ik} a_{jk}$$

If $C = A A^T$

then $$= \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{pmatrix}$$

$$C_{ij} = \sum_k a_{ik} a_{jk} \quad \leftarrow \quad \mathrm{Cov\ of\ } X_i, X_j$$

Covariance matrix of $\begin{matrix} X_1 \\ \vdots \\ X_m \end{matrix}$

- C is $m \times m$
- is symmetric
- semi-positive definite

The joint pdf of $X_1, \ldots, X_m$

$$\Rightarrow f(x_1, \ldots, x_m) = \frac{1}{(2\pi)^{m/2}\sqrt{\det C}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T C^{-1}(\vec{x}-\vec{\mu})}$$

Let $m = n$.

pdf for $Z_1, \ldots, Z_m$.

$$f(z_1, \ldots z_m) = f(z_1) \cdots f(z_m)$$

$$= \frac{1}{(2\pi)^{m/2}} e^{-\frac{1}{2}(z_1^2 + z_2^2 + \ldots + z_m^2)}$$

$$= \frac{1}{(2\pi)^{m/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}}$$

$$\int_S \frac{1}{(2\pi)^{m/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}} \, dz_1 \cdots dz_m$$

and $\vec{z}^T\vec{z} = \vec{x}^T C^{-1}\vec{x}$

$= \vec{x}^T (AA^T)^{-1}\vec{x}$

$= \vec{x}^T A^{-T} A^{-1}\vec{x}$

$= (A^{-1}\vec{x})^T (A^{-1}\vec{x})$

Let $\vec{z} = A^{-1}\vec{x}$

$A\vec{z} = \vec{x}$