Theory of Probability

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The Weak Law of Large Numbers (WLLN):

For a collection of random variables $X_1, X_2, \dots$ which are IID

with $E[X_i] = \mu < \infty$

$Var[X_i] = \sigma^2 < \infty$,

if $\epsilon > 0$, then

$$P\left[\left|\frac{1}{n}\sum X_i - \mu\right| \geq \epsilon\right] \to 0 \quad \text{as} \quad n \to \infty.$$

$$Var\left(\frac{1}{n}\sum X_i\right) = \frac{\sigma^2}{n} \to 0 \quad \text{as} \quad n \to \infty.$$

The Central Limit Theorem (CLT)

If $X_1, X_2, \dots$ are IID random variables with

$E[X_i] = \mu < \infty$

$Var[X_i] = \sigma^2 < \infty$

then

$$Y_n = \underbrace{\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\sigma/\sqrt{n}}}_{\substack{E = 0 \\ Var = 1}} \rightsquigarrow N(0,1) \quad \text{as} \quad n \to \infty.$$

Ie. $\lim_{n\to\infty} P[Y_n \leq a] = \Phi(a)$. CDF of $N(0,1)$.

Lemma Let $X_1, X_2, \ldots$ be a sequence of random variables with CDFs F_{X_i} and MGFs M_{X_i}, for $i = 1, 2, 3, \ldots$. Let X have CDF F_X and MGF M_X.

If $\lim_{n\to\infty} M_{X_n}(t) = M_X(t)$ for all t,

then $\lim_{n\to\infty} F_{X_n}(t) = F_X(t)$ for all t at which F_X is continuous.

Proof of CLT:

Assume that $\mu = 0$, $\sigma^2 = 1$.

Let $M(t) = E\left[e^{tX_i}\right]$.

Then $E\left[e^{tX_i/\sqrt{n}}\right] = M\left(\frac{t}{\sqrt{n}}\right)$.

and $E\left[e^{t\sum_{i=1}^{n} X_i/\sqrt{n}}\right] = \left(M\left(t/\sqrt{n}\right)\right)^n$.

$$\left(\text{If } \mu = 0,\ \sigma^2 = 1,\quad \frac{\frac{1}{n}\sum X_i - \mu}{\sigma/\sqrt{n}} = \frac{\frac{1}{n}\sum X_i}{1/\sqrt{n}} = \frac{1}{\sqrt{n}}\sum X_i\right)$$

The calculation is slightly easier if we set

$$L(t) = \log M(t).$$

Note $L(0) = \log M(0) = \log 1 = 0$

$$L'(t) = \frac{1}{M(t)} M'(t) \implies L'(0) = \frac{M'(0)}{M(0)} = M'(0) = E[X_i] = 0$$

and $$L''(t) = \frac{M(t)\,M''(t) - M'(t)\,M'(t)}{M(t)^2}$$

$$= \frac{M(t)\,M''(t) - (M'(t))^2}{(M(t))^2}$$

$$L''(0) = \frac{M(0)\,M''(0) - (M'(0))^2}{(M(0))^2}$$

$$= \frac{1 \cdot E[X^2] - 0}{1} = E[X^2] = 1$$

To prove the CLT, show that

$$\lim_{n\to\infty} \left(M(t/\sqrt{n})\right)^n = e^{t^2/2}$$

or equivalently:

$$\lim_{n\to\infty} \underbrace{n \cdot \log M(t/\sqrt{n})}_{n\,L(t/\sqrt{n})} = t^2/2$$

Just compute:

$$\lim_{n\to\infty} n\,L(t/\sqrt{n}) = \lim \frac{L(t/\sqrt{n})}{1/n}$$

by L'Hopital

$$= \lim \frac{-L'(t/\sqrt{n})\,\frac{1}{2}\,t/n^{3/2}}{-1/n^2}$$

$$= \lim \frac{t\,L'(t/\sqrt{n})}{2/\sqrt{n}}$$

by L'Hopital

$$= \lim \frac{\cancel{\not} t\, L''(t/\sqrt{n})\, \cancel{\tfrac{1}{2}}\, \cancel{\tfrac{1}{n^{3/2}}}\, t}{\cancel{\not}\, 2 \cdot \cancel{\tfrac{1}{2}}\, 1/\cancel{n^{3/2}}}$$

$$= \lim_{n \to \infty} \frac{t^2}{2} L''(t/\sqrt{n})$$

$$= t^2/2$$

$$\Rightarrow \lim_{n \to \infty} \log M(t/\sqrt{n})^n = t^2/2$$

$$\Leftrightarrow \lim_{n \to \infty} M(t/\sqrt{n})^n = e^{t^2/2}$$

$$\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \to N(0,1) \quad \text{as} \quad n \to \infty. \quad \blacksquare$$

Another heuristic method:

$$\lim_{n \to \infty} (M(t/\sqrt{n}))^n = e^{t^2/2}$$

Expand $M(t/\sqrt{n})$ in a Taylor series:

$$M(t/\sqrt{n}) = M(0) + \cancel{\frac{1}{\sqrt{n}} M'(0)}\, t + \frac{1}{n} M''(0) \frac{t^2}{2} + \ldots$$

$$= 1 + \frac{1}{n} t^2/2 + \ldots$$

$$M(t/\sqrt{n})^n = \left(1 + \frac{t^2/2}{n} + \ldots\right)^n$$

And we know that $\lim_{n \to \infty} \left(1 + \frac{t}{n}\right)^n = e^t$.

Ex: Student's t-distribution

PDF: $\sim \left(1 + \frac{x^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}$