Theory of Probability

Dec 7, 2020

Strong Law of Large Numbers:

For IID random variables $X_1, X_2, X_3, \ldots$ with $E[X_i] = \mu < \infty$, then

$$P\left[\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = \mu\right] = 1.$$

Weak Law of Large Numbers

$$\lim_{n\to\infty} P\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \epsilon\right] = 0 \quad \text{for any } \epsilon > 0.$$

Assumption for proof: $Var[X_i] = \sigma^2 < \infty$.

Assumption for proof: $E[X_i^4] = K < \infty$.

Proof of the SLLN:

WLG we can assume that $E[X_i] = 0$.

Let $S_n = \sum_{i=1}^{n} X_i$.

$(X_1 + .. X_n)(X_1 + .. + X_n) \cdot$

$(X_1 + .. + X_n)(X_1 + .. + X_n)$

Compute: $E[S_n^4] = E\left[\left(X_1 + X_2 + .. + X_n\right)^4\right]$

since independent and $E[X_i] = 0$.

$$= E\left[\sum X_i^4 + \cancel{\sum X_i^3 X_j} + \sum X_i^2 X_j^2 + \cancel{\sum X_i^2 X_j X_k} + \cancel{\sum X_i X_j X_k X_l}\right]$$

$$= E\Big[\sum X_i^4 + \sum_{i \neq j} X_i^2 X_j^2 \Big]$$

$$= n\,E[X_1^4] + \binom{4}{2}\binom{n}{2} \underbrace{E[X_1^2 X_2^2]}_{\text{independent}}$$

$$= nK + 6\,\frac{n(n-1)}{2}\,\underbrace{E[X_1^2]\,E[X_2^2]}_{(E[X_1^2])^2}$$

Now note that : $\mathrm{Var}[X_i^2] = E[X_i^4] - (E[X_i^2])^2 \geq 0$

$$\Rightarrow (E[X_1^2])^2 = E[X_1^2]\,E[X_2] \leq K = E[X_1^4].$$

$$\Rightarrow E[S_n^4] \leq nK + 3n(n-1)K$$

$$\Rightarrow \underbrace{E\Big[\frac{S_n^4}{n^4}\Big]}_{\left(\frac{S_n}{n}\right)^4} \leq \frac{K}{n^3} + \frac{3(n^2-n)K}{n^4}$$

$$\leq \frac{K}{n^3} + \frac{3K}{n^2} - \frac{3K}{n^3}$$

$$\leq \frac{K}{n^3} + \frac{3K}{n^2}$$

$$\Rightarrow E\Big[\frac{S_n^4}{n^4}\Big] \leq \frac{K}{n^3} + \frac{3K}{n^2}$$

$$\Rightarrow E\Big[\sum_{n=1}^{\infty} \frac{S_n^4}{n^4}\Big] = \sum_{n=1}^{\infty} E\Big[\frac{S_n^4}{n^4}\Big] \leq \sum_{n=1}^{\infty}\Big(\frac{K}{n^3} + \frac{3K}{n^2}\Big) < \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{S_n^4}{n^4} < \infty \text{ with probability } 1.$$

And therefore, with probability 1,

the $\lim_{n\to\infty} \underbrace{\frac{S_n^4}{n^4}}_{\text{the } n^{th} \text{ term in } \sum_1^\infty \frac{S_n^4}{n^4}.} = 0.$

And therefore, with probability 1,

$$\lim_{n\to\infty} \frac{S_n}{n} = 0.$$

$$\downarrow$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i$$