Lecture 2

September 14th 2020

1 Multinomial coefficients

Multinomial coefficient is an analogous of binomial coefficient in that they correspond directly to the counting problems.
Imagine we have N distinct objects:

$$0_1, 0_2, ..., 0_N$$

sort into M bins: N_i objects into bin i (Order of each bin does <u>not</u> matter)

$$\sum_{i=1}^{M} n_i = N$$

$$\binom{N}{n_1} * \binom{N-n_1}{n_2} * \binom{N-n_1-n_2}{n_3} * ... * \binom{N-n_1-...-n_{M-1}}{n_M}$$

(Filling bin 1) * (Filling bin 2) * (Filling bin 3) * ... * (Filling bin M)

$$= \frac{N!}{n_1!(N-n_1)!} * \frac{(N-n_1)!}{n_2!(N-n_1-n_2)!} * \frac{(N-n_1-n_2)!}{n_3!(N-n_1-n_2-n_3)!} * ... * \frac{n_M}{n_M! * 1!}$$

$$= \frac{N!}{n_1!n_2!n_3!...n_M!} = \binom{N}{n_1, n_2, n_3, ..., n_M} (Multinomial\ Coefficient)$$

Multinomial Coefficient is the number of ways to sort N distinct objects into bins, each having n_i objects.

2 Extension to Multinomial Theorem

$$(x_1 + x_2 + ... + x_r)^n = \Sigma \binom{n}{(n_1, n_2, ..., n_r)} * x_1^{n_1} * x_2^{n_2} * ... * x_r^{n_r}$$

Here, Σ is sum over $n_1, n_2, ..., n_r$ such that $n_1 + n_2 + ... + n_r = n$

3 Sample Spaces and Events

Sample space (S) is the set of possible outcomes of an experiment.

Ex: Roll two dice

$$S = \{(1,1),(2,1),(3,1),...,(6,6)\}$$

A collection of outcomes is called an event. An event is a subset of S.

Ex: Let E and F be two events defined on the sample space S.
1-1) Define a new event $G = E \cup F$ where E is a set of the first die = 2, and F is a set of the second die less than 3.

Here, $\cup$ is all outcomes that are contained in E or F.

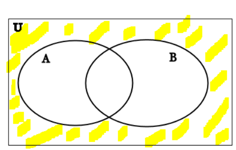
$E = \{(2,1),(2,2),...,(2,6)\}$
$F = \{(1,1),(2,1),(3,1),...(6,1),(1,2),(2,2),...,(6,2)\}$
Therefore, $G = E \cup F = \{(2,1),(2,2),...,(2,6),(1,1),(1,2),(3,1),(3,2),...,(6,1),(6,2)\}$

1-2) Define another new event $H = E \cap F$

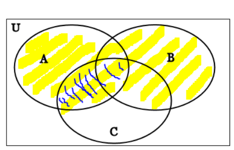
Here, $\cap$ is outcomes that are contained in both E and F.

Therefore, $H = E \cap F = \{(2,1),(2,2)\}$

Definition: Mutually Exclusive
We say E and F are mutually exclusive if
$E \cap F = \{\}$ (empty set)
$= \emptyset$ (the null set)
Other properties of set theory useful for probability
Multiple events: $G = \cup_{i=1}^{n} E_i$
Complement: E^c = all outcomes that are not in E



Here, $U = E \cup E^c, E \cap E^c = \emptyset$



Here, Yellow part is: $A \cup B$, Blue part is: $A \cap C$

4 Set Operation Laws

- **Commutative Law**: $E \cup F = F \cup E$, $E \cap F = F \cap E$

- **Associative Law**: $(E \cup F) \cup G = E \cup (F \cup G)$, $(E \cap F) \cap G = E \cap (F \cap G)$

- **Distributive Law**: $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$

4.1 DeMorgan's Laws

DeMorgan's Law: the following laws could be combined into DeMorgan's Laws as such:
$(\cup_{i=1}^{n} E_i)^c = \cap_{i=1}^{n} E_i^c$
$(\cap_{i=1}^{n} E_i)^c = \cup_{i=1}^{n} E_i^c$

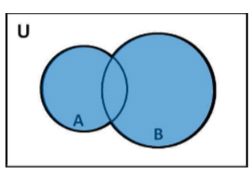
Hint:

- Multinomial Coefficient: number of ways to group n objects into groups of n_1,n_2,..,n_r object, with $n_1 + n_2$+...+$n_r = n$.

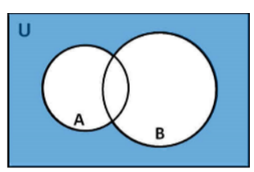
- Binomial Coefficient: number of ways to group n objects into groups of n_1,n_2 object, with $n_1 = k$ and $n_2 = n - k$.

5 Graphical representation of DeMorgan's laws

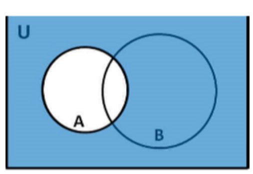
1. $(\cup_{i=1}^{n} E_i)^c = \cap_{i=1}^{n} E_i^c$



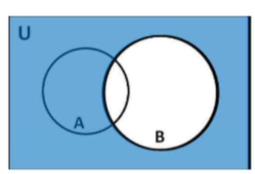
$A \cup B$



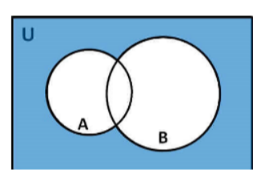
$(A \cup B)^c$



A^c

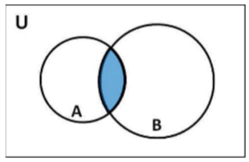


B^c

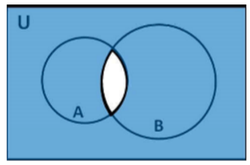


$A^c \cap B^c$

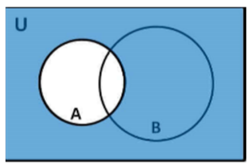
2. $(\cap_{i=1}^{n} E_i)^c = \cup_{i=1}^{n} E_i^c$



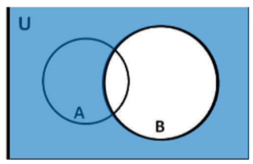
$A \cap B$



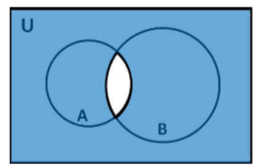
$(A \cap B)^c$



A^c



B^c



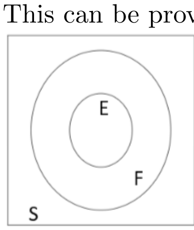
$A^c \cup B^c$

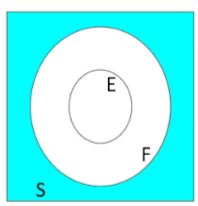
6 Example problems

Example 1

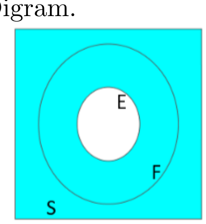
Prove that if $E \subset F$, then $F^c \subset E^c$.

This can be proved with a Venn Digram.





F^c

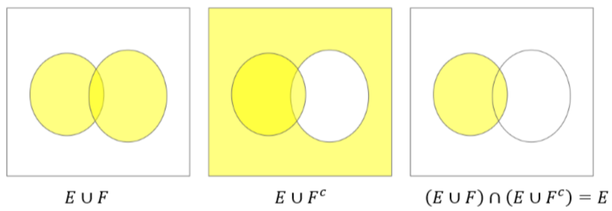


E^c

We can see that $F^c \subset E^c$

Example 2

Simplify $(E \cup F)(E \cup F^c)$ **using a Venn Digram**



We can see that the common area is exactly E

7 Textbook Example:

1.Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one league and the B team in another. How many different divisions are possible?

Solution: There are $\frac{10!}{5!5!} = 252$ possible divisions.

2.In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

Solution: Note that this example is different from the previous example because now the order of the two teams is irrelevant. That is, there is no A and B team, but just a division consisting of 2 groups of 5 each. Hence, the desired answer is $\frac{10!/(5!5!)}{2!} = 126$.