Lecture Preparation Sep 28th

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September 28th, 2020

1 Lecture Notes

Recall that

$$P(E|F) = \frac{P(EF)}{P(F)}$$

If F does not affect $E|F$, then E and F are independent events. This means that

$$P(E|F) = P(E) = \frac{P(EF)}{P(F)}$$

Definition: E and F are independent events if

$$P(EF) = P(E)P(F)$$

If $P(EF) \neq P(E)P(F)$, then E, F are dependent events.

Example: Roll 2 dice. $E =$ sum is 6, $F =$ die 1 is 4.

$$P(E) = 5/36, P(F) = 1/6, P(EF) = 1/36$$

$$P(E)P(F) = (5/36)(1/6) = 5/216 \neq 1/36$$

Therefore, E and F are dependent.

Example: Roll 2 dice. $G =$ sum is 7, $F =$ die 1 is 4.

$$P(G) = 6/36 = 1/6, P(F) = 1/6, P(GF) = 1/36$$

$$P(G)P(F) = (1/6)(1/6) = 1/36 = 1/36$$

Therefore, E and F are independent.

1.1 Proposition 4.1

If E, F are independent, then so are E and F^c.

$$P(E) = P(EF) + P(EF^c) = P(E)P(F) + P(EF^c)$$

$$P(EF^c) = P(E) - P(E)P(F) = P(E)(1 - P(F)) = P(E)P(F^c)$$

Question: If E, F are independent, and E, G are independent, is E independent of the event FG?

Example: Roll 2 dice. $E =$ sum is 7, $F =$ die 1 is 4, G second die is 3.
From the previous example,

$$P(EF) = P(E)P(F)$$

$$P(EG) = P(E)P(G)$$

$$P(E|FG) = 1 \neq P(E) = 1/6$$

1.2 Definition

The events E, F, G are independent if

$$P(EFG) = P(E)P(F)P(G)$$

$$P(EF) = P(E)P(F)$$
$$P(EG) = P(E)P(G)$$
$$P(FG) = P(F)P(G)$$

If these conditions hold, then E is independent of any function of E, G.

Example: Toss a coin N times, the outcome of each toss is H or T and independent of all other tosses. Each toss is known as a trial.

If A, B are independent, we have shown that

$$P(A|B) = P(A) = \frac{P(AB)}{P(B)}$$

This means that the ratio of the area of A to that of the entire sample space S is the same as the ratio of the area AB to that of B.

2 In class Examples

2.1 Example 1

Independent Trials: An infinite series of trials, each has probability of p of success, and failure q = 1-p

What is the probability there is at least one success in the first n trials?

$$P(\text{at least one success}) = 1 - P(\text{no success at all}) = 1 - P(F1F2F3...Fn) = 1 - P(F1)P(F2)P(F3) = 1 - (1 - P(F)^n)$$

What is the probability that of exactly k success in the first n trials?

$$\binom{n}{k} P(S)^k * P(F)^{n-k}$$

Probability of all n trials success?

$$P(S)^n]$$

2.2 Example 2

A list of independent events with probability of success P. What is the probability of n success before m failures?

Pascal's Solution:

Let E be the events that n success before m failures

$$P(E) = \mathbf{P}n, m(\text{n success before m failures}) = P(EF) + P(EF^c)$$

where F = event trial 1 is a success

=P(success on trial one $\cap$n-1 subsequent successes before m failures) + P(failure on trial 1 $\cap$ n success before m-1 failures)

=p Pn-1, m + (1-p) Pn,m-1

This is called **Recurrence relation**

Fermat's Solution:

If n success occur before m failures, there have to be n successes in the first n+m-1 trials.

$$P(\text{k success in n+m-1 trials}) = \binom{m+n-1}{k} * p^k(1-p)^{m+n-1-k}$$

Pn,m = $\sum_{k=n}^{m+n-1} \binom{m+n-1}{k} p^k(1-p)^{m+n-1-k}$

3 Additional Examples

3.1 problem1 (Ex 4g)

A system composed of n separate components is said to be a parallel system if it functions when at least one of the components functions. For such a system, if component i, which is independent of the other components, functions with probability pi , i = 1, . . . , n, what is the probability that the system functions?

P(system functions) = 1 - P (system does not function)
=1 - P($\bigcap_i A_i^c$)
=1 - P($\prod_{i=1}^{n}(1 - pi)$

3.2 problem2

Suppose that initially there are r players, with player i having ni units, ni > 0, i = 1, . . . , r. At each stage, two of the players are chosen to play a game, with the winner of the game receiving 1 unit from the loser. Any player whose fortune drops to 0 is eliminated, and this continues until a single player has all n =

$$sum_{i=1}^{r} ni$$

units, with that player designated as the victor. Assuming that the results of successive games are independent and that each game is equally likely to be won by either of its two players, find Pi, the probability that player i is the victor?

Solution:

To begin,suppose that there are n players, with each player initially having 1 unit. Consider player i. Each stage she plays will be equally likely to result in her either winning or losing 1 unit, with the results from each stage being independent. In addition, she will continue to play stages until her fortune becomes either 0 or n. Because this is the same for all n players, it follows that each player has the same chance of being the victor, implying that each player has probability 1/n of being the victor. Now, suppose these n players are divided into r teams, with team i containing ni players, i = 1, . . . , r. Then the probability that the victor is a member of team i is ni/n. But because

(a) team i initially has a total fortune of ni units, i = 1,...,r, and

(b) each game played by members of different teams is equally likely to be won by either player and results in the fortune of members of the winning team increasing by 1 and the fortune of the members of the losing team decreasing by 1,

it is easy to see that the probability that the victor is from team i is exactly the prob- ability we desire. Thus, Pi = ni/n. Interestingly, our argument shows that this result does not depend on how the players in each stage are chosen.

In the gambler's ruin problem, there are only 2 gamblers, but they are not assumed to be of equal skill.