Conditional probability Section3.5

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1 Lecture Video

This lecture is mainly a revisit to the topic of conditional probability, and we will be discussing why conditional probability can be considered as just a normal probability for a fixed even that we are conditioned in.
First, let us review the definition of conditional probability.

Definition (Conditional probability) *For two event $E, F \subset S$, the conditional probability of E given F is*

$$P(E|F) = \frac{P(EF)}{D(F)}$$

Now we fix the event F, and consider the function

$$Q(E) = P(E|F)$$

It turns out that $Q = P(X(\text{some event})|F)$ satisfies the axioms of probability.

Axioms of probability:

1. $0 \leq Q(E) \leq 1$

   Proof. $0 \leq \frac{P(EF)}{P(F)} \leq 1 \implies 0 \leq \underbrace{P(EF) \leq P(F)}_{\text{because } EF \subseteq F}$ □

2. $P(S(\text{sample space})|F) = 1 = Q(S)$

   Proof. $P(S|F) = \frac{P(SF)}{P(F)} = \frac{P(F)}{P(F)} = 1$ □

3. Let E_i be mutually exclusive events, then $Q(\bigcup_i E_i) = \sum_i Q(E_i)$.

   Proof. $P(\bigcup E_i|P) = \dfrac{P(\bigcup E_i \cap F)}{P(F)} = \dfrac{P(\bigcup E_i F)}{P(F)} = \dfrac{\sum P(E_i F)}{P(F)} = Q(E_i)$
   Where each $E_i F$ are mutually exclusive. □

Since $Q(E) = P(E|F)$ defines a new probability function. Consider conditional probabilities under Q:

$$\begin{aligned} Q(E_1|E_2) &= \frac{Q(E_1E_2)}{Q(E_2)} = \frac{P(E_1E_2|F)}{P(E_2|F)} \\ &= \frac{P(E_1E_2F)}{P(F)} \cdot \frac{P(F)}{P(E_2F)} = \frac{P(E_1E_2F)}{P(E_2F)} \end{aligned}$$

1.1 Conditional Independence

Definition (Conditional independence) *E_1 and E_2 are conditionally independence with respect to F if*

$$P(E_1|E_2F) = P(E_1|F)$$

Which is very similar to the definition of independence: $P(A|B) = P(A)$

An equivalent definition of conditional independence is

$$P(E_1E_2|F) = P(E_1|F)P(E_2|F)$$

Compare with $P(AB) = P(A)P(B)$ if A and B are independent.

Proof. To prove the equivalence, first we have

$$\begin{aligned} P(E_1|E_2F) &= \frac{P(E_1E_2F)}{P(E_2F)} \\ &= \frac{P(E_1E_2|F)P(F)}{P(E_2F)} \end{aligned}$$

Since $P(E_1|E_2F) = P(E_1F)$, we then have that

$$P(E_1|F) = \frac{P(E_1E_2|F)P(F)}{P(E_2F)}$$

Now back to $P(E_1E_2|F)$, we have

$$P(E_1|E_2F) = P(E_1|F)\frac{P(E_2F)}{P(F)}$$

Since $\dfrac{P(E_2F)}{P(F)} = P(E_2|F)$ by definition, thus finally

$$P(E_1E_2|F) = P(E_1|F)P(E_2|F)$$

□

2 In Class Examples

Ex 3.33(Self-test):
Let E, F, G, be independent events, show that $P(E|FG^c) = P(E)$.

By independence, $P(EFG^c) = P(E)P(F)P(G^c)$. Similarly, $P(FG^c) = P(F)P(G^c)$

$$\begin{aligned} P(E \mid FG^c) &= \frac{P(EFG^c)}{P(FG^c)} \\ &= \frac{P(E)P(F)P(G^c)}{P(F)P(G^c)} \\ &= P(E) \end{aligned}$$

Ex 3.35(Self-test)
If 4 balls are randomly chosen from an urn containing 4 red, 5 white, 6 blue, and 7 green balls, find the conditional probability they are all white given that all balls are of the same color.

Let S = event of drawing balls of the same color
Let W = event of drawing all white balls

$$\begin{aligned} P(W) &= \binom{5}{4} / \binom{22}{4} \\ P(S) &= (1 + \binom{5}{4} + \binom{6}{4} + \binom{7}{4}) / \binom{22}{4} \\ P(W \mid S) &= \frac{P(W \cap S)}{P(S)} \\ &= \frac{P(W)}{P(S)} \\ &= \binom{5}{4} / (1 + \binom{5}{4} + \binom{6}{4} + \binom{7}{4}) \\ &= 5/56 \end{aligned}$$

Ex 3.27(Self-test)
An urn initially contains 1 red and 1 blue ball. At each stage, a ball is randomly withdrawn and replaced by two other balls of the same color.(For instance, if the red ball is initially chosen, then there would be 2 red and 1 blue balls in the urn when the next selection occurs.) Show by mathematical l induction that the probability that there are exactly i red balls in the urn after n stages have been completed is $\frac{1}{n+1}$, $1 \le i \le n+1$.

Using Mathematical Induction:
1.Show that the result is true for stage n=0,1,2

Stage 0	1R,1B			
Stage 1	2R,1B	1R,2B		
Stage 2	3R,1B	2R,2B	2R,2B	1R,3B

Let R_n = the number of red balls in stage n.
Let B_n = the number of blue balls in stage n.
Let R = the event of drawing a red ball.
Let B = the event of drawing a blue ball.

Stage 0: $\frac{1}{n+1} = 1$
$P(i = 1) = 1$

Stage 1: $\frac{1}{n+1} = P(R_0 = 1 \cap B) = 1/2$
$P(i = 1) = P((R_0 = 1) \cap B) = 1/2$
$P(i = 2) = P((R_0 = 1) \cap R) = 1/2$

Stage 2: $\frac{1}{n+1} = 1/3$
$P(i = 1) = P((R_1 = 1) \cap B) = 1/2 * 2/3 = 1/3$
$P(i = 2) = P((R_1 = 1) \cap R) + P((R_1 = 2) \cap B) = 1/2 * 1/3 + 1/2 * 1/3 = 1/3$
$P(i = 3) = P((R_1 = 2) \cap R) = 1/2 * 2/3 = 1/3$

2.Assuming the result holds after stage n, show the result is true for stage $n+1$.

Stage $n + 1$: $\frac{1}{n+1} = 1/(n + 2)$
Observe that for stage n, the urn contains a total of $n + 2$ balls.
$P(i = i) = P((R_n = i) \cap B) + P((R_n = i - 1) \cap R) = \frac{n+2-i}{n+1} * \frac{1}{n+1} + \frac{i-1}{n+2} * \frac{1}{n+1} = \frac{n+2-i+i-1}{(n+2)(n+1)} = \frac{n+1}{(n+2)(n+1)} = \frac{1}{n+2}$

The proof is complete.

3 Additional Problems

Here are some examples from the textbook (*Ross, A First Course in Probability, 10th Ed., 2018*):

Example 1: (self-test 3.11) A type C battery is in working condition with probability .7, whereas a type D battery is in working condition with probability .4. A battery is randomly chosen from a bin consisting of 8 type C and 6 type D batteries.

(a) What is the probability that the battery works?

(b) Given that the battery does not work, what is the conditional probability that it was a type C battery?

Solution:

(a) Let C denotes the event that the battery we randomly picked from the bin is a type C battery, D denotes the event that the battery we randomly picked from the bin is a type D battery, and E denoted that the battery we randomly picked from the bin works.

Now, we know that:

$$
\begin{aligned}
P(C) &= \text{the battery we randomly picked from the bin is a type C battery} \\
&= \frac{8}{8+6} = \frac{4}{7} \\
P(D) &= \text{the battery we randomly picked from the bin is a type D battery} \\
&= \frac{6}{8+6} = \frac{3}{7} \\
P(E|C) &= \text{the probability that a type C battery is in working condition} \\
&= 0.7 \\
P(E|D) &= \text{the probability that a type D battery is in working condition} \\
&= 0.4 \\
P(E) &= P(EC) + P(ED) \qquad \{\text{The battery we picked is either a type C or a type D}\} \\
&= P(E|C)P(C) + P(E|D)P(D) \\
&= 0.7 \times \frac{4}{7} + 0.4 \times \frac{3}{7} \\
&= \frac{4}{7}
\end{aligned}
$$

(b)

$$
\begin{aligned}
P(C|E^c) &= \frac{P(C \cap E^c)}{P(E^c)} \\
&= \frac{P(E^c|C)P(C)}{P(E^c)} \qquad \{\text{Bayes' Rule}\} \\
&= \frac{\Big[1 - P(E|C)\Big]P(C)}{1 - P(E)} \\
&= \frac{0.3 \times \frac{4}{7}}{1 - \frac{4}{7}} \\
&= \frac{2}{5}
\end{aligned}
$$

Example 2: (self-test 3.14) A coin having probability .8 of landing on heads is flipped. A observes the result—either heads or tails—and rushes off to tell B. However, with probability .4, A will have forgotten the result by the time he reaches B. If A has forgotten, then, rather than admitting this to B, he is equally likely to tell B that the coin landed on heads or that it landed tails. (If he does remember, then he tells B the correct result.)

(a) What is the probability that B is told that the coin landed on heads?

(b) What is the probability that B is told the correct result?

(c) Given that B is told that the coin landed on heads, what is the probability that it did in fact land on heads?

Solution:

(a) Let A denotes the event that A remembers the coin result, B denotes the event that B is told the coin landed on heads, H denotes the event that the coin was actually landed on head.

$$\begin{aligned} P(B) &= P(BA) + P(BA^c) \qquad \{\text{Law of total probability}\} \\ &= P(H|A)P(A) + P(B|A^c)P(A^c) \\ &= 0.8 \times 0.6 + 0.5 \times (1 - 0.6) \\ &= 0.68 \end{aligned}$$

(b) Let C denotes the event that B is told the correct result.

$$\begin{aligned} P(C) &= P(BH) + P(B^cH^c) \\ &= P(BHA) + P(BHA^c) + P(B^cH^cA) + P(B^cH^cA^c) \\ &= P(BH|A)P(A) + P(BH|A^c)P(A^c) + P(B^cH^c|A)P(A) + P(B^cH^c|A^c)P(A^c) \\ &= P(BH|A)P(A) + P(B|A^c)P(H|A^c)P(A^c) + P(B^cH^c|A)P(A) \\ &\quad + P(B^c|A^c)P(H^c|A^c)P(A^c) \\ &= 0.8 \cdot 0.6 + 0.5 \cdot 0.8 \cdot 0.4 + 0.2 \cdot 0.6 + 0.5 \cdot 0.2 \cdot 0.4 \\ &= 0.8 \end{aligned}$$

(c)

$$P(H|B) = \frac{P(BH)}{P(B)}$$

We've already computed P(BH) at part(b)
and P(B) at part (a)

$$= \frac{0.8 \cdot 0.6 + 0.5 \cdot 0.8 \cdot 0.4}{0.68}$$

$$= \frac{0.64}{0.68}$$

$$= \frac{16}{17}$$

Example 3: (self-test 3.30) For any events E and F, show that $P(E|E \cup F) \geq P(E|F)$.

<u>Solution</u>:

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{P(EF)}{P(EF) + P(E^cF)}$$

$$P(E|E \cup F) = \frac{P(E)}{P(E \cup F)} = \frac{P(EF^c) + P(EF)}{P(EF^c) + P(EF) + P(E^cF)}$$

Observe that when computing $P(E|E \cup F)$, you are adding the same constant, $P(EF^c)$ to the numerator and denominator of $P(E|F)$. By the first axiom of probability, we know $P(E|F) \leq 1$. When you add the same constant to a fraction less than or equal to one, the numerator effect dominates. In percentage term, the numerator increased more since it is smaller than or equal to the denominator. Therefore, the resulting value, in this case, $P(E|E \cup F)$, will always be greater than or equal to the starting value,$P(E|F)$.