Math 233 Note

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1 Lecture Video

Expectation of X:

Definition: E[x]=$\sum_{i=1} x_i p(x_i)$ where $P[X = x_i]p(x_i)$.

Consider the following example:

Rolling a die and we get double the amount.(i.e. roll 1 get 2, roll 2 get 4) So $Y = x^2$ if Y is the amount we get and X is the number we roll.

$$P[Y=1]=P[X=1]=\frac{1}{6}$$
$$P[Y=4]=P[X=2]=\frac{1}{6}$$
$$\vdots$$
$$E[Y]=\sum_i y_i p_y(y_i)=\sum_i x_i^2 p_x(x_i)$$

In general, we can show that if Y=g(x), then:

E[Y]=E[g(x)]=$\sum_i g(x_i)p(x_i)$.

Note: $E[x^2] \neq (E[x])^2$

$(E[x])^2 = (\sum_{i=1} x_i p(x_i))^2 = \sum_i \sum_j x_i x_j p(x_i)p(x_j) \neq \sum_i x_i^2 p(xi)$

Corollary:Expectation is a linear operation:

$$E[aX+b]=\sum_i (ax_i + b)p(x_i)$$
$$=a\sum_i x_i p(x_i) + b\sum_i p(x_i)$$
$$=aE[X]+b \text{ (E is a linear transformation)}$$

Remark:E[x] is also known as the 1^{st} moment of X.
Furthermore, $E[x^n]$ is known as the n^{th} moment of X.

$$E[X^n] = \sum_i x_i^n p(x_i)$$

Variance:It's useful to talk about characteristics of random variables:
1.Exptected Value (mean)
2.Max Value, Min Value
3.Mode (Most likely value)
4."Spread" of a random variable

Consider the following Random Variables:

-P[W=0]=1 E[W=0]

-P[Y=-1]=$\frac{1}{2}$ P[Y=1]=$\frac{1}{2}$ E[Y]=0

-P[Z=-100]=$\frac{1}{2}$ P[Z=100]=$\frac{1}{2}$ E[Y]=0

Now:
Let u=E[x]

We want to characterize $E[|x-u|]$, but it turns out to be better mathematically to examine $E[(x-u)^2]$.

$$\text{Variance of X=Var[X]=}E[(x-u)^2]\text{=}E[(x-E[x])^2)$$

$$E[(x-u)^2] = \sum (x_i - u)^2 p(x_i)$$

$$=\sum (x_i^2 - 2ux_i + u^2)p(x_i)$$

$$=\sum xi^2 p(x_i) - 2u \sum x_i p(x_i) + u^2 \sum p(x_i)$$

$$=E[x^2] - 2u^2 + u^2$$

$$=E[x^2] - (E[X])^2$$

Note:On the other hand, variance is not a linear transformation.

$$Var[aX+b] = E[(aX+b)^2] - (E[aX+b])^2$$

$$=E[a^2X^2 + 2abX + b^2] - (au+b)^2$$

$$=a^2E[x^2] + 2abu + b^2 - a^2u^2 - 2abu - b^2$$

$$=a^2(E[X^2] - (E[X])^2)$$

$$=a^2Var[X]$$

Another useful (common) quantity is the standard deviation:

$$\text{std[X]=}\sqrt{Var[X]}$$

$$\text{std[aX+b]=}\sqrt{Var[aX+b]}$$

$$=\sqrt{a^2Var[X]}$$

$$\text{=astd[X]}$$

2 Related Examples

1.Let X denote a random variable that takes on any of the values 1, 0, and 1 with respective probabilities.Compute E[X^2].

$$P[X=-1]=P[X=1]=p[X=0]=\frac{1}{3}$$

Let Y=X^2.

$$P[Y=1]=P[X=1 \text{ or } X=-1]=\frac{2}{3}$$

$$p[Y=0]=p[X=0]=\frac{1}{3}$$

$$E[X^2] = E[Y] = 0\frac{1}{3}+1\frac{2}{3}=\frac{2}{3}$$

2.Calculate Var(X) if X represents the outcome when a fair die is rolled.

Solution:

We already know that the E[x]=3.5.

Also:

$$E[X^2] = 1^2\frac{1}{6}+2^2\frac{1}{6}+3^2\frac{1}{6}+4^2\frac{1}{6}+5^2\frac{1}{6}+6^2\frac{1}{6}=\frac{1}{6}(91)$$

Hence,

$$Var(X)=\frac{91}{6}-(\frac{7}{2})^2 = 35/12$$

Note: A useful identity is that, for any constants a and b, Var(aX+b)=$a^2 Var(X)$

Remarks:

(a)Analogous to the means being the center of gravity of a distribution of mass, the variance represents, in the terminology of mechanics, the moment of inertia.

(b)The square root of the Var(X) is called the standard deviation of X, and we denote it by SD(X).

3.Five fair coins are flipped. If the outcomes are assumed independent, find the probability mass function of the number of heads obtained.

Solution: If we let X equal the number of heads (successes) that appear, then X is a binomial random variable with parameters (n = 5, p = $\frac{1}{2}$). Hence,

$$P[X=0]=\binom{5}{0}+(\tfrac{1}{2})^0+(\tfrac{1}{2})^5=\tfrac{1}{32}$$

$$P[X=1]=\binom{5}{1}+(\tfrac{1}{2})^1+(\tfrac{1}{2})^4=\tfrac{1}{32}$$

$$P[X=2]=\binom{5}{2}+(\tfrac{1}{2})^2+(\tfrac{1}{2})^3=\tfrac{1}{32}$$

$$P[X=3]=\binom{5}{3}+(\tfrac{1}{2})^3+(\tfrac{1}{2})^2=\tfrac{1}{32}$$

$$P[X=4]=\binom{5}{4}+(\tfrac{1}{2})^4+(\tfrac{1}{2})^1=\tfrac{1}{32}$$

$$P[X=5]=\binom{5}{5}+(\tfrac{1}{2})^5+(\tfrac{1}{2})^0=\tfrac{1}{32}$$

4. Suppose that a particular trait (such as eye color or left-handedness) of a person is classified on the basis of one pair of genes, and suppose also that d represents a dominant gene and r a recessive gene. Thus, a person with dd genes is purely dominant, one with rr is purely recessive, and one with rd is hybrid. The purely dominant and the hybrid individuals are alike in appearance. Children receive 1 gene from each parent. If, with respect to a particular trait, 2 hybrid parents have a total of 4 children, what is the probability that 3 of the 4 children have the outward appearance of the dominant gene?

Solution: If we assume that each child is equally likely to inherit either of 2 genes from each parent, the probabilities that the child of 2 hybrid parents will have *dd*, *rr*, and *rd* pairs of genes are, respectively, $\frac{1}{4}$, $\frac{1}{4}$, and $\frac{1}{2}$.

Hence, since an offspring will have the outward appearance of the dominant gene if its gene pair is either *dd* or *rd*, it follows that the number of such children is binomially distributed with parameters (4, $\frac{3}{4}$). Thus, the desired probability is

$$\binom{4}{3}(\tfrac{3}{4})^3(\tfrac{1}{4})^1=\tfrac{27}{64}$$

5. Suppose that

$$P[X = a] = p, P[X = b] = 1 \;\; p$$

(a) Show that $\frac{X-b}{a-b}$ is a Bernoulli random variable.

(b) Find Var(X).

Solution: Since $\frac{X-b}{a-b}$ will equal 1 with probability p or 0 with probability (1-p), it follows that it is a Bernoulli random variable with parameter p. Because the variance of such a Bernoulli random variable is p(1-p), we have

$$p(1\text{-}p) = \mathrm{Var}(\tfrac{X-b}{a-b}) = \tfrac{1}{(a-b)^2}\mathrm{Var}(X\text{-}b) = \tfrac{1}{(a-b)^2}\mathrm{Var}(X)$$

Hence,

$$\mathrm{Var}(X) = (a\text{-}b)^2 p(1-p)$$

6.There are 2 coins in a bin. When one of them is flipped, it lands on heads with probability .6, and when the other is flipped, it lands on heads with probability .3. One of these coins is to be randomly chosen and then flipped. Without knowing which coin is chosen, you can bet any amount up to 10 dollars, and you then either win that amount if the coin comes up heads or lose it if it comes up tails. Suppose, however, that an insider is willing to sell you, for an amount C, the information as to which coin was selected. What is your expected payoff if you buy this information? Note that if you buy it and then bet x, you will end up either winning x C orx C (that is,losing x + C in the latter case). Also, for what values of C does it pay to purchase the information?

Solution:If you wager x on a bet that wins the amount wagered with probability p and loses that amount with probability 1 p, then your expected winnings are:

$$xp - x(1 - p)=(2p - 1)x$$

which is positive (and increasing in x) if and only if p is greater than $\frac{1}{2}$. Thus, if p is less than or equal to 1/2, one maximizes one's expected return by wagering 0, and if p is greater than $\frac{1}{2}$, one maximizes one's expected return by wagering the maximal possible bet. Therefore, if the information is that the .6 coin was chosen, then you should bet 10, and if the information is that the .3 coin was chosen, then you should bet 0. Hence, your expected payoff is

$$\frac{1}{2}(1.2 - 1)10 + \frac{1}{2}0=1 - c$$

Since your expected payoff is 0 without the information (because in this case the probability of winning is $\frac{1}{2}$ (.6) + $\frac{1}{2}$(.3) less than (1/2), it follows that if the information costs less than 1, then it pays to purchase it.