Student Notes

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19 October 2020

1 Notes from Recorded Lecture and Textbook

a) The Poisson Random Variable

X˜Poisson(λ)

$P[X=k]=\frac{e^{-\lambda}\lambda^k}{k!}$(Poisson probability mass function)

> *Key Approximation:* Poisson is approximately the same as binomial (n,p) when n is large and p is small, and λ=np is $\theta(1)$

If X˜binomial(n,p), then

$$\begin{aligned} P[X=k] &= \binom{n}{k}p^k(1-p)^{n-k} && (1)\\ &= \frac{n!}{k!(n-k)!}(\frac{\lambda}{n})^k(1-\frac{\lambda}{n})^{n-k} && (2)\\ &= \frac{n!}{(n-k)!n^k}\frac{\lambda^k}{k!}\frac{(1-\frac{\lambda}{n})^n}{(1-\frac{\lambda}{n})^k} && (3)\\ &= \frac{n(n-1)(n-2)(n-3)...(n-k+1)}{\underbrace{n\cdot n\cdot n...\cdot n}_{n^k}}\frac{\lambda^k}{k!}\frac{(1-\frac{\lambda}{n})^n}{(1-\frac{\lambda}{n})^k} && (4) \end{aligned}$$

as $n\to\infty$

$\frac{n(n-1)(n-2)(n-3)...(n-k+1)}{\underbrace{n\cdot n\cdot n...\cdot n}_{n^k}}\approx 1 \qquad (1-\frac{\lambda}{n})^n=e^{-\lambda} \qquad (1-\frac{\lambda}{n})^k\approx 1$

$$\approx\frac{e^{-\lambda}\lambda^k}{k!} \qquad (5)$$

b) Expected Value and Variance

If X˜Binomial(n,p)

E[X]=np, Var[X]=np(1-p),

> *Conjecture:* If n is large, p is small, λ=np is $\theta(1)$,
>
> then E[Y]$\approx E[X]=$$\lambda$
>
> Var[Y]$\approx Var[X]=np(1-p)=$$\lambda$(1-p)$\approx$$\lambda$

Expected Value:

$$E[Y] = \sum_{k=0}^{\infty} \frac{ke^{-\lambda}\lambda^k}{k!} \tag{6}$$
$$= \sum_{k=1}^{\infty} \frac{ke^{-\lambda}\lambda^k}{k!} \tag{7}$$
$$= e^{-\lambda}\sum_{k=1}^{\infty} \frac{k\lambda^k}{k!} \tag{8}$$
$$= e^{-\lambda}\sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \tag{9}$$
$$= e^{-\lambda}\lambda\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \tag{10}$$

$$\because \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = e^{\lambda}$$

$$\therefore E[Y] = e^{-\lambda}\lambda e^{\lambda} \tag{11}$$
$$= \lambda \tag{12}$$

Variance:

$$E[Y^2] = \sum_{k=0}^{\infty} \frac{k^2e^{-\lambda}\lambda^k}{k!} \tag{13}$$
$$= \sum_{k=1}^{\infty} \frac{k^2e^{-\lambda}\lambda^k}{k!} \tag{14}$$
$$= \lambda\sum_{k=1}^{\infty} \frac{ke^{-\lambda}\lambda^{k-1}}{(k-1)!} \tag{15}$$
$$= \lambda\sum_{k=0}^{\infty} \frac{(k+1)e^{-\lambda}\lambda^k}{k!} \tag{16}$$
$$= \lambda[\sum_{k=0}^{\infty} \frac{ke^{-\lambda}\lambda^k}{k!} + \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!}] \tag{17}$$

$$\because \sum_{k=0}^{\infty} \frac{ke^{-\lambda}\lambda^k}{k!} = E[Y] = \lambda \quad and \quad \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1$$

$$\therefore E[Y^2] = \lambda(\lambda+1) \tag{18}$$

$$Var[Y] = E[Y^2] - (E[Y])^2 \tag{19}$$
$$= \lambda(\lambda+1) - \lambda^2 \tag{20}$$
$$= \lambda \tag{21}$$

c) Poisson Paradigm

Weakly Dependent: If P[E|F]=P[E], then E and F are independent. If P[E|F]≈P[E], then E and F are ***weakly dependent***

Poisson Paradigm:

"For the number of events to occur to approximately have a Poisson distribution, it is not essential that all the events have the same probability of occurrence, but only that all of these probabilities be small. The following is referred to as the Poisson paradigm" (textbook section 4.7).

Consider n trials with $P[E_i]=p_i$. If n is large, all the p_is are small, and either the E_is are ***independent*** or ***weakly dependent***, then the sum $E=E_1+E_2+E_3...+E_n$ approximately has a Poisson distribution with parameter

$$\lambda = \sum_{i=1}^{n} p_i$$

(instead of λ=np, we will get λ=np if p_is are the same).

d) Rare Events Modelling

Rare Events: Events that happen with some possibility but rarely happen consecutively/at the same time/together within a short time interval. e.g. There may be a Hurricane in Louisiana, but it is very unlikely to have 2 Hurricanes on the same day in Louisiana.

We can use Poisson random variable to approximate the number of rare events happening in a large time interval **[0,t]** by dividing the large time interval into **n** pieces of small time intervals.

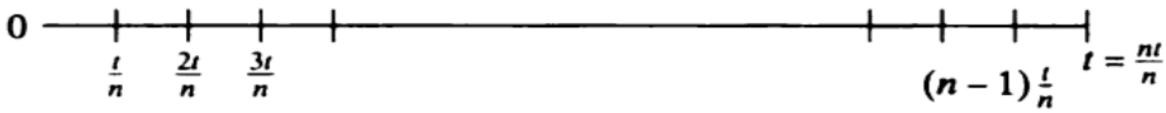

Figure 1: from textbook section 4.7

Assume 1) n is large enough so that either 1 or 0 events happen in each interval.
2) Independence between intervals–"What ever occurs in one interval has no (probability) effect on what will occur in other nonoverlapping intervals" (textbook section 4.7).
Show that the number of events occurring within the large time interval t is a Poisson random variable with parameter λt.

"*Assumption 1:* The probability that 1 event occurs in a given interval of length h is equal to $\lambda h + o(h)$, where $o(h)$ stands for any function f(h) that is such that

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

(h=width of the subinterval=$\frac{t}{n}$)

Assumption 2: The probability that 2 or more events occur in an interval of length h is equal to o(h).

Assumption 3: For any integers $n, j_1, j_2, ...j_n$, and any set of n nonoverlapping intervals, if we define E_i to be the event that exactly j_i of the events under consideration occur in the ith of these intervals, then events $E_1, E_2, E_3, ..., E_n$ are independent" (textbook section 4.7).

N(t) is the number of events in the large time interval t.

We would like to obtain an expression for P[N(t)=k]

P[N(t)=k]=P[k of n subintervals have 1 event and other (n-k) sub-intervals have 0 event] + P[N(t)=k and at least 1 subinterval has 2 or more events]

-Let A denote events k of n subintervals having 1 event and other (n-k) subintervals having 0 events.

-Let B denote events N(t)=k and at least 1 subinterval having 2 or more events.

P[N(t)=k]=P[A]+P[B]

Find P[B]

$$\begin{aligned}
P[B] &\leq P[\text{at least 1 subinterval has 2 or more events}] && (22)\\
&= P\left(\bigcup_{i=1}^{n} [\text{ith subinterval has 2 or more events}]\right) && (23)\\
&\leq \sum_{i=1}^{n} P[\text{ith subinterval has 2 or more events}] && (24)\\
&= \sum_{i=1}^{n} o(\frac{t}{n}) \qquad (from\ assumption\ 2) && (25)\\
&= no(\frac{t}{n}) && (26)\\
&= t\frac{o(\frac{t}{n})}{\frac{t}{n}} && (27)\\
P[B] &= 0\ as\ n \to \infty && (28)
\end{aligned}$$

($\because$ when n$\to \infty$, $\frac{t}{n} \to 0$, and $\because$ $\frac{o(h)}{h} = 0\ as$ h$\to 0$, $\therefore$ $\frac{o(\frac{t}{n})}{\frac{t}{n}} = 0$)

Find P[A]

P[ith subinterval has 0 events]

$$\begin{aligned}
&= 1 - P[\text{1 event in ith interval}] - P[\text{2 or more events in ith interval}] && (29)\\
&= 1 - \lambda h - o(h) - o(h) = 1 - \lambda h - o(h) \qquad (from\ assumptions\ 1, 2, also\ 2o(h) = o(h)) && (30)
\end{aligned}$$

$$P[A] = P[\text{k of n subintervals have 1 event and other (n-k) subintervals have 0 event}] \tag{31}$$
$$= \binom{n}{k}(P[\text{ith subinterval has 1 event}])^k (P[\text{ith subinterval has 0 event}])^{n-k} \tag{32}$$
$$= \binom{n}{k}\Big[\frac{\lambda t}{n} + o\Big(\frac{t}{n}\Big)\Big]^k \Big[\frac{\lambda t}{n} - o\Big(\frac{t}{n}\Big)\Big]^{n-k} \tag{33}$$

$$as\ n \to \infty, \quad n\Big[\frac{\lambda t}{n} + o\Big(\frac{t}{n}\Big)\Big] = \lambda t + t\Big[\frac{o(\frac{t}{n})}{\frac{t}{n}}\Big] \to \lambda t$$

$$P[A] = \binom{n}{k}\Big(\frac{\lambda t}{n}\Big)^k \Big(1 - \frac{\lambda t}{n}\Big)^{n-k} \tag{34}$$

Use Poisson to approximate the binomial.
as n$\to \infty$, P(A)$\approx \frac{e^{-\lambda t}(\lambda t)^k}{k!}$

$$P[N(t) = k] = P[A] + P[B] = \frac{e^{-\lambda t}(\lambda t)^k}{k!} + 0 = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

2 Notes from Zoom Lecture

Recap: A random variable X that takes on one of the values 0,1,2,.... is said to be a Poisson random variable with parameter λ if, for some $\lambda > 0$

$$P[X = k] = e^{-\lambda}\frac{\lambda^k}{k!}$$
$$E(X) = \lambda$$
$$Var[X] = \lambda$$

In Class Examples:

1. Theoretical Exercise 4.17:

Let X be a Poisson random variable with parameter λ. Show that P$[X = i]$ increases monotonically and then decreases monotonically as i increases, reaching its maximum when i is the largest integer not exceeding λ

Solution:

Show that $\frac{P[X=k+1]}{P[X=k]}$>1

$$\frac{P[X = k+1]}{P[X = k]} = \frac{e^{-\lambda}\frac{\lambda^{k+1}}{(k+1)!}}{e^{-\lambda}\frac{\lambda^k}{k!}} = \frac{\lambda}{k+1}$$
$$\frac{\lambda}{k+1} > 1 \text{ if } k < \lambda - 1$$

$P[X = i]$ increases monotonically and then decreases monotonically as i increases, reaching its maximum when i is the largest integer not exceeding λ

2. Self-test Exercise 4.14:

On average, 5.2 hurricanes hit a certain region every year. What is the probability that there will be 3 or fewer hurricanes this year?

Solution:

Let X = number of hurricanes this year, then X˜Poisson(λ), and $\lambda = 5.2$

$$\begin{aligned} P[X \leq 3] &= P[X = 0] + P[X = 1] + P[X = 2] + P[X = 3] && (35) \\ &= \sum_{k=0}^{3} e^{-\lambda} \frac{\lambda^k}{k!} && (36) \\ &\approx 24\% && (37) \end{aligned}$$

3. Self-test exercise 4.15:

The number of eggs laid on a tree leaf by an insect of a certain type is a Poisson random variable with parameter λ. However, such a random variable can be observed only if it is positive, since if it is 0, then we cannot know that such an insect was on the leaf. If we let Y denote the observed number of eggs, then

$$P[Y = i] = P[X = i | X > 0]$$

where X is Poisson with parameter λ. Find $E[Y]$

Solution:

$$\begin{aligned} E[Y] &= \sum_{k=1}^{\infty} kP[Y = k] && (38) \\ &= \sum_{k=1}^{\infty} kP[X = k | X > 0] && (39) \\ &= \sum_{k=1}^{\infty} k \frac{P[(X = k) \cap (X > 0)]}{P[X > 0]} && (40) \\ &= \sum_{k=1}^{\infty} k \frac{P[X = k]}{1 - P[X = 0]} && (41) \\ &= \frac{1}{1 - e^{-\lambda}} \sum_{k=1}^{\infty} kP[X = k] \qquad (Notice\, that P[X = 0] = e^{-\lambda} \frac{\lambda^0}{0!} = e^{-\lambda}) && (42) \\ &= \frac{1}{1 - e^{-\lambda}} \sum_{k=0}^{\infty} kP[X = k] && (43) \\ &= \frac{E[X]}{1 - e^{-\lambda}} && (44) \\ &= \frac{\lambda}{1 - e^{-\lambda}} && (45) \end{aligned}$$

3 Extra Exercises

1. Theoretical Exercises 4.19:

Let X be a Poisson random variable with parameter λ, what value of λ maximize $P[X = k], k \geq 0$?

Solution:

$$P[X = k] = \frac{e^{-\lambda} \lambda^k}{k!}$$

Taking logarithms on both sides:

$$\ln(P[X=k]) = \ln(\frac{e^{-\lambda}\lambda^k}{k!}) \tag{46}$$
$$= \ln(e^{-\lambda}) + \ln(\lambda^k) - \ln(k!) \tag{47}$$
$$= -\lambda + k\ln(\lambda) - \ln(k!) \tag{48}$$

Differentiating the above equation with respect to λ:

$$\frac{\partial}{\partial\lambda}\ln(P[X=k]) = \frac{\partial}{\partial\lambda}[-\lambda + k\ln(\lambda) - \ln(k!)] \tag{49}$$
$$= -1 + \frac{k}{\lambda} - 0 \tag{50}$$

Equating the above to 0: λ = k maximize P[X=k]

2. Theoretical Exercises 4.20:

Show that X is a Poisson random variable with parameter λ, then

$$E[X^n]{=}\lambda E[(X+1)^{n-1}]$$

Now use this result to compute $E[X^3]$

Solution:

Given that $X\tilde{}$Poisson(λ), then $P[X=x] = e^{-\lambda}\frac{\lambda^x}{x!}$, x=0,1,2....

$$E[X^n] = \sum_{x=0}^{\infty} x^n \frac{e^{-\lambda}\lambda^x}{x!} \tag{51}$$
$$= \sum_{x=1}^{\infty} \frac{x^n e^{-\lambda}\lambda^x}{x!} \tag{52}$$
$$= \lambda\sum_{x=1}^{\infty} \frac{x^n e^{-\lambda}\lambda^{x-1}}{x!} \tag{53}$$
$$= \lambda\sum_{x=1}^{\infty} x^{n-1}\frac{\lambda^{x-1}}{(x-1)!}e^{-\lambda} \tag{54}$$
$$= \lambda\sum_{x=0}^{\infty} (x+1)^{n-1}\frac{\lambda^{x}}{x!}e^{-\lambda} \tag{55}$$
$$= \lambda E[(X+1)^{n-1}] \tag{56}$$

Now we have that

$$E(X^3) = \lambda E[(X+1)^2] \tag{57}$$
$$= \lambda E[X^2 + 2X + 1] \tag{58}$$
$$= \lambda(E[X^2] + 2E[X] + 1) \tag{59}$$
$$= \lambda(\lambda E[X+1] + 2E[X] + 1) \tag{60}$$
$$= \lambda(\lambda E[X] + \lambda + 2\lambda + 1) \tag{61}$$
$$= \lambda(\lambda^2 + 3\lambda + 1) \tag{62}$$
$$= \lambda^3 + 3\lambda^2 + \lambda \tag{63}$$