Student Note - Theory of Probability Lecture 15

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1 Uniform Random Variable

1.1 Basics

The uniform random variable is one that takes on value in an interval with equal probability: **U**$\sim$ **Uniform**(α, β)

[PDF] $f_U(u) = \begin{cases} \dfrac{1}{\beta - \alpha}, \text{ for } u \in (\alpha, \beta) \\ 0, \text{ otherwise} \end{cases}$

[CDF] $F_U(u) = \begin{cases} 0, \text{ for } u \leq \alpha \\ \dfrac{u-\alpha}{\beta - \alpha}, \text{ for } u \in (\alpha, \beta) \\ 1, \text{ for } u \geq \beta \end{cases}$

$$=> \mathrm{P}[u \in (a,b)] = \int\limits_a^b \frac{1}{\beta - \alpha} du = \frac{b-a}{\beta - \alpha}, \text{ for } (a,b) \subset (\alpha, \beta)$$

[Moments]

$\mathrm{E}[U] = \int\limits_\alpha^\beta u \; du = \frac{\alpha+\beta}{2}$

$\mathrm{Var}[U] = \int\limits_\alpha^\beta (u - \frac{\alpha+\beta}{2})^2 \; du = \frac{(\alpha-\beta)^2}{12}$

1.2 Example: U $\sim$ Uniform$(0, 1)$

[PDF] $f_U(u) = \begin{cases} \dfrac{1}{1-0} = 1, \text{ for } u \in (0,1) \\ 0, \text{ otherwise} \end{cases}$

[CDF] $F_U(u) = \begin{cases} 0, \text{ for } u \leq 0 \\ \dfrac{u-0}{1-0} = u, \text{ for } u \in (0,1) \\ 1, \text{ for } u \geq 1 \end{cases}$

[Moments] $\mathrm{E}[U] = \frac{1+0}{2} = \frac{1}{2}$, $\mathrm{Var}[U] = \frac{(1-0)^2}{12} = \frac{1}{12}$

2 Normal Random Variable

2.1 Basics

$\mathbf{X} \sim (\mu, \sigma^2)$, for $x \in (-\infty, \infty)$

[PDF] $f_X(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

[CDF]$F_X(x) = \int\limits_{-\infty}^{x} f(y)dy = \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$

2.2 Important Properties

2.2.1 Operations on Normal Random Variable

If $\mathbf{X} \sim (\mu, \sigma^2)$, then $Y = aX + b \sim \mathbf{Norm}(a\mu + b, a^2\mu^2)$

Proof:

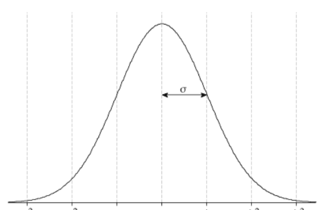
P[Y$\leq y] = P[aX + b \leq y] = P[X \leq \frac{y-b}{a}] = F_x(y - b/a) \Rightarrow F_Y(y) = F_X[\frac{y-b}{a}]$

$\frac{d}{dy} P[Y \leq y] = \frac{d}{dy} F_X[\frac{y-b}{a}] = \frac{1}{a} f_X[\frac{y-b}{a}] = \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma}^{-\frac{(\frac{y-b}{a} - \mu)^2}{2\sigma^2}}$

$= \frac{1}{\sqrt{2\pi} \cdot \sigma}^{\frac{[y-(a \cdot \mu + b)]^2}{2a^2\mu^2}} = \frac{1}{\sqrt{2\pi} \cdot \sigma}^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}$

This shows the PDF of $Y \sim \mathbf{Norm}(a\mu + b, a^2\mu^2)$

2.2.2 Graph: Bell Curve



1. The mean(μ) controls the **centre** of the distribution
2. The variance(σ^2) controls the **spread** of the distribution

Source: http://amsi.org.au/ESA_Senior_Years/SeniorTopic4/4f/4f_2content_3.html

2.3 Standard Normal Distribution

2.3.1 Standard Normal Variable: $Z \sim Norm[0, 1]$

[PDF] $f(z) = \frac{1}{\sqrt{2\pi}} * e^{-\frac{e^2}{2}}$

[CDF] $F(z) = \int\limits_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} * e^{-\frac{e^2}{2}} dt = \Phi(z)$

2.3.2 Standardization of Normal Random Variable

If $X \sim Norm(\mu, \sigma^2)$, then we can standardize it as

$$Z = \frac{X - \mu}{\sigma} \sim Norm(0, 1)$$

Therefore, any normal random variable can be converted into a standard normal random variable.

2.4 Application: Modeling Continuous Return Rate

If $\begin{cases} S_0 = \text{price of stock today} \\ S_1 = \text{price of stock tomorrow} \end{cases}$

S_1 is often modeled as:

$$S_1 = S_0 * e^r, \quad r = \text{continuous return rate}$$

where $\ln(\frac{S_1}{S_0}) = r \sim N(0, 1)$, which is a standard normal random variable

[**Remark**] Return rates (r) is usually modeled as **standard normal**Als Stock prices (S_t) is usually modeled as **standard log-normal** random variables.

2.5 The Normal Approximation of Binomial Distribution

DeMoivre-Lapalace Limit Theorem: For X ~ Binomial(n,p), when **n(n-p)** is large, we can use a standard normal random variable to approximate a standardized binomial variable:
If $S_n \sim \text{Binomial}(n, p)$, then:

$$\lim_{n\to\infty} [\frac{S_n - np}{\sqrt{np \cdot (1-p)}} \in (a, b)] = \phi(b) - \phi(a) = \int_a^b \frac{1}{\sqrt{2\pi} \cdot e^{-\frac{t^2}{2}}} \, dz$$

[**Remark**]

1. $\phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi} \cdot e^{-\frac{t^2}{2}}} \, dy$

2. When p is fixed, the distribution of $\frac{S_n - np}{\sqrt{np \cdot (1-p)}}$ gets closer to standard normal distribution as n gets larger.

3. This approximation is useful when n is too large such that the calculation of $\binom{n}{k}$ numerically difficult.

[Cookbook]

When np(1-p) is large enough.

$P[X = k] = P[k - \frac{1}{2} < X < k + \frac{1}{2}]$ **(continuity correction)**

$$= P[\frac{k-\frac{1}{2}-np}{np\cdot(1-p)} < X < \frac{k+\frac{1}{2}-np}{np\cdot(1-p)}]$$

$$\approx \phi[\frac{k+\frac{1}{2}-np}{np\cdot(1-p)}] - \phi[\frac{k-\frac{1}{2}-np}{np\cdot(1-p)}]$$