Probability Lecture Student Notes

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Asynchronous Lecture

Expectations of Sums

Recall that:

- In the discrete case, $E[X] = \sum_i x_i p(x_i)$,
- In the continuous case, $E[X] = \int x f(x) dx$

If random variables X, Y have joint pdf $f(x, y)$, then

$$E[g(X,Y)] = \int\int g(x,y)f(x,y)dxdy$$

$g(X, Y)$ may be anything from X, to Y, to $2XY^2$.

Example: $g(x, y) = x + y$

$$\begin{aligned} E[g(X,Y)] &= \int\int (x+y)f(x,y)dxdy \\ &= \int\int xf(x,y)dxdy + \int\int yf(x,y)dxdy \\ &= \int xf_x(x)dx + \int yf_y(y)dy \\ &= E[X] + E[Y] \end{aligned}$$

Linearity of expectation $\Rightarrow E[X_1+X_2+\cdots+X_n] = E[X_1]+E[X_2]+\cdots+E[X_n]$

Note: this is not necessarily the case for infinite sums. Only if the following exchange of $\lim_{n\to\infty}$ and $\sum_{i=1}^{n} x_i$ holds is it true.

$$\begin{aligned} E[\sum_{i=1}^{\infty} x_i] &= E[\lim_{n\to\infty} \sum_{i=1}^{n} x_i] \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} x_i E[x_i] \end{aligned}$$

Guaranteed to be exchangeable if $\forall i, x_i \geq 0$; or if $\sum_{i=1}^{n} E[|x_i|] < 0$ (absolutely convergent)

Covariance and Variance of Sums

Definition: *Covariance*

$$\begin{aligned} Cov(X,Y) &= E[(X-\mu_x)(Y-\mu_y)] \\ &= E[XY] - E[X]E[Y] \\ &= \int\int xyf(x,y)dxdy - E[X]E[Y] \end{aligned}$$

If X and Y are independent, then $Cov(X,Y) = 0$
Note: this is not necessarily true in reverse!

Covariance Properties

1. $Cov(X,Y) = Cov(Y,X)$
2. $Cov(X,X) = Var(X)$
3. $Cov(aX,Y) = aCov(X,Y)$
4. $Cov(\sum_{i=1}^{n} x_i, \sum_{j=1}^{m} y_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} Cov(x_i, y_j)$

If we combine properties 4 and 2,

$$\begin{aligned} Var(\sum_{i=1}^{n} x_i) &= Cov(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} Cov(x_i, x_j) \end{aligned}$$

We can pull out where $x_i = x_j$, which occurs when $i = j$

$$= \sum_{i=1}^{n} Var(x_i) + \sum_{i \neq j} Cov(x_i, x_j)$$

And then add property 1,

$$= \sum_{i=1}^{n} Var(x_i) + 2 \sum_{i<j} Cov(x_i, x_j)$$

If the x_i are independent, $Var(\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} Var(x_i)$

Correlation

Definition: *Correlation*

$$\rho(x, y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

We can compare this to linear algebra. Recall that the dot (innder) product $\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i$, and that the norm $|\vec{x}| = \vec{x} \cdot \vec{x}$. We can also rewrite the dot product as

$$\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}| \cos\theta$$

and replace each term with parts from the definition of covariance, such that

$$\vec{x} \cdot \vec{y} \to Cov(X, Y) \qquad |\vec{x}| \to std(x)$$
$$|\vec{x}|^2 \to var(x) \qquad \cos\theta \to \rho(x, y)$$

Then, we end up with

$$\cos\theta = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|}$$

If $\rho(x, y) = 0$, we say that x and y are *uncorrelated*,
or in vector terms, they are *orthogonal* ($\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$)
If $\rho(x, y) = 1$, we know that $x = aX + b$,
or in vector terms, they are *colinear* ($\cos\theta = 1 \Rightarrow \theta = 0$)

Synchronous Lecture

Expectations of Sums

Recall that for a collection of Random Variables $X_1 + X_2 + ... + X_n$, if $Y = X_1 + X_2 + ... + X_n$, then $E[X] = E[X_1 + X_2 + ... + X_n = \sum E[X_i]$

Definition: *Sample mean*

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

If X_i are IID Random Variables, with each $E[x_i] = \mu_i$,

$$\begin{aligned} E[\bar{x}] &= \frac{1}{n}(\sum E[X_i]) \\ &= \frac{1}{n} \cdot \mu \\ &= \mu \end{aligned}$$

note: in statistics, by observing data, we infer property of the R.V.process. For here, we estimate the mean of those Random Variables by taking average of data.

$$\begin{aligned} E[g(X, Y)] &= \int\int (x + y) f(x, y) dx dy \\ &= \int\int x f(x, y) dx dy + \int\int y f(x, y) dx dy \\ &= \int x f_x(x) dx + \int y f_y(y) dy \\ &= E[X] + E[Y] \end{aligned}$$

Covariance

Definition: ***Covariance***

$$Cov(X, Y) = E[(X - \mu_x)(Y - \mu_y)]$$

If X, Y independent, then $Cov(X, Y) = 0$

Correlation

Definition: ***Correlation***

$$\rho(x, y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

with $\rho(x, y) \in [-1, 1]$.

If we consider $Cov(aX, Y)$,

$$\begin{aligned} Cov(aX, Y) &= aCov(X, Y) \\ \rho(aX, Y) &= \frac{aCov(X, Y)}{\sqrt{var(aX)}\sqrt{var(Y)}} \\ &= \frac{\cancel{a}Cov(X, Y)}{\sqrt{\cancel{a^2}var(X)}\sqrt{var(Y)}} \\ &= \frac{a}{|a|}\rho(X, Y) \end{aligned}$$

note: correlation shows that we scale the random variables by some characteristic size, and then we compute Covariances.

Sample Variance

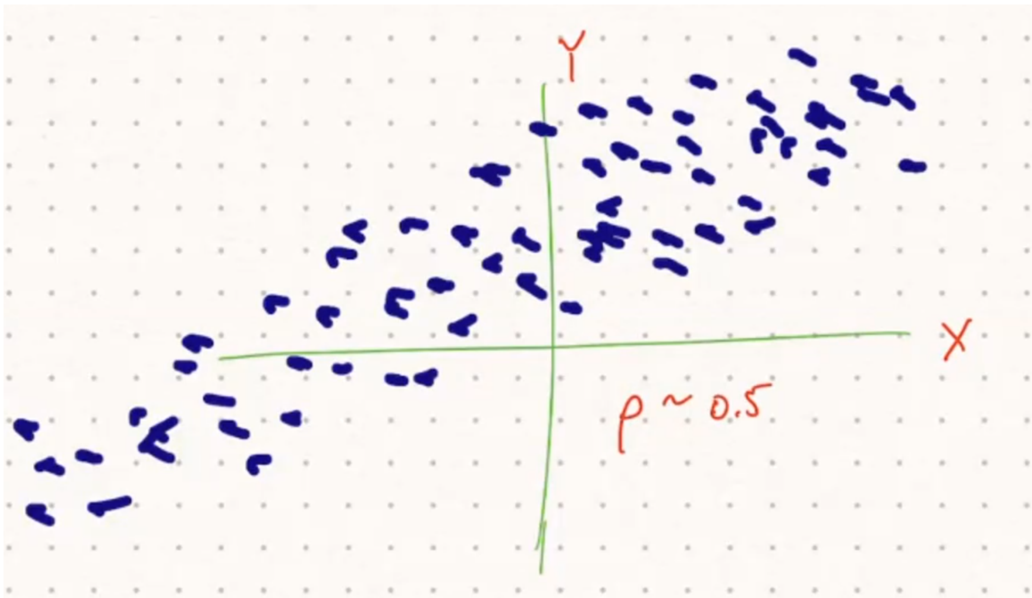
Definition: ***Sample Variance***

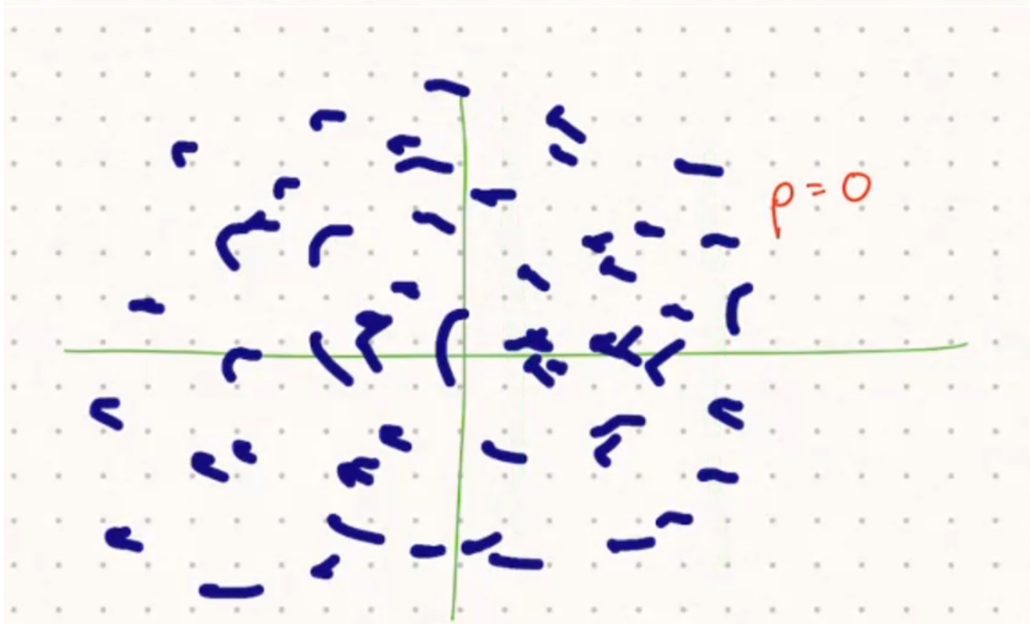
If $X_1 + X_2 + \cdots + X_n$ are IID Random Variables with mean μ and variance σ^2, then the sample variance is given by

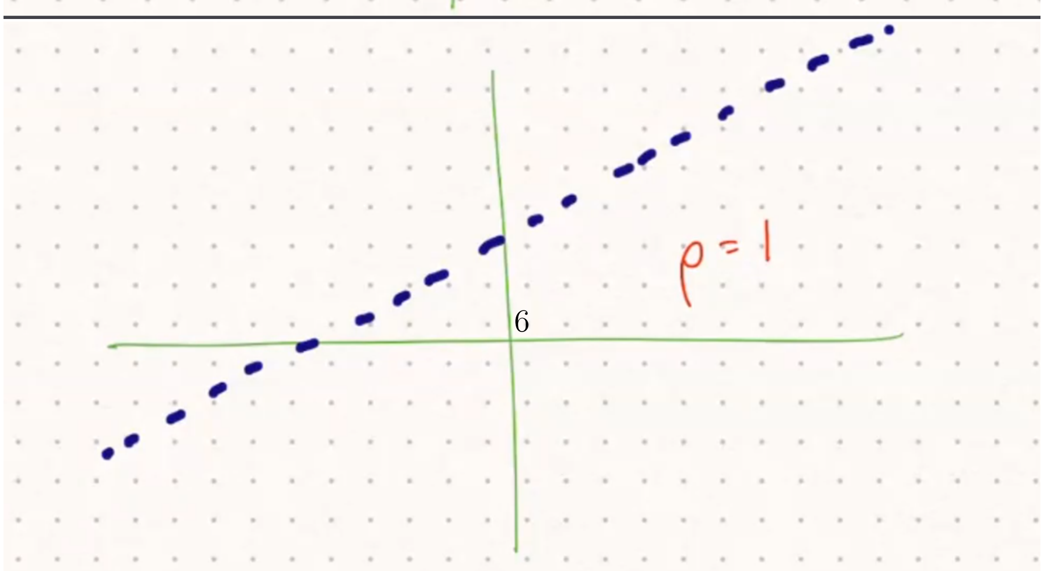
$$S^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n - 1}$$

Note that $E[S^2] = \sigma^2$.

Student question about how to visualize these computations. There are three graphs of visualization of different covariances.


Y
X
ρ ~ 0.5


ρ = 0


ρ = 1

Exercises

Theoretical exercise 7.23

If $Y = a + bX$, what is $\rho(X, Y)$?

$$\begin{aligned} Var(X) &= \sigma^2 \\ Var(Y) &= Var(a + bX) \\ &= b^2\sigma^2 \end{aligned}$$

$$\begin{aligned} Cov(X, Y) &= Cov(X, a + bX) \\ &= E(X - \mu)[\cancel{a} + bX - (\cancel{a} + b\mu)] \\ &= E[(X - \mu)(bX - b\mu)] \\ &= bE[(X - \mu)(X - \mu)] \\ &= b\sigma^2 \end{aligned}$$

$$\begin{aligned} \rho(X, Y) &= \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \\ &= \frac{b\sigma^{\cancel{2}}}{\cancel{\sigma} \cdot |b|\sigma} \\ &= \frac{b}{|b|} \end{aligned}$$

1 if $b > 0$, and -1 if $b < 0$

Theoretical exercise 7.4

Let X be a Random Variable with

$$\begin{aligned} E[X] &= \mu < \infty \\ Var[X] &= \sigma^2 < \infty \end{aligned}$$

and $g = g(x)$ is twice differentiable, how to approximate $E[g(x)]$?

By Taylor Series, we know that

$$\begin{aligned}
g(x) &\approx g(\mu) + g'(\mu)(x-\mu) + \frac{g''(\mu)}{2}(x-\mu)^2 \\
E[g(x)] &\approx E[g(\mu) + g'(\mu)(x-\mu) + \frac{g''(\mu)}{2}(x-\mu)^2] \\
&= g(\mu) + g'(\mu)E[X-\mu] + \frac{g"(\mu)}{2}E[(X-\mu)^2] \\
&= g(\mu) + \frac{g"(\mu)}{2}\sigma^2
\end{aligned}$$

$\underline{E[g(x)] \approx g(\mu) + \frac{g"(\mu)}{2}\sigma^2}$