Moment Generating Functions 7.7
Properties of Normal Random Variables 7.8

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1 Moment Generating Functions (Ross 7.7)

Moment Generating Function $M(t)$ is

$$M(t) = E[e^{tX}]$$

$$= \begin{cases} \sum_x e^{tx} p(x) & \text{if X is discrete with mass function } p(x) \\ \int e^{tx} f(x)\, dx & \text{if X is continuous with density } f(x) \end{cases}$$

where t is a number, X is a random variable, and $f(x)$ is a probability density function of X.
(Note: The function $M(t)$ for $t \neq 0$ might not exist)

We can think of the moment generating function $M_X(t)$ as a map from the probability density function $f(x)$ to a new function with e over the region where $f(x)$ is not zero.

If two functions have the same moment generating function, then they must have the same distribution.

We call $M(t)$ the moment generating function because all of the moments of X can be obtained by successively differentiating $M(t)$ and then evaluating the result at $t = 0$.
For example:

$$M(0) = E[e^0] = E[1] = 1$$

The 1^{st} derivative of M(t) is

$$\begin{aligned} M'(t) &= \frac{\partial}{\partial t} \int e^{tx} f(x)\, dx \\ &= \int \frac{\partial}{\partial t} e^{tx} f(x)\, dx \\ &= \int x e^{tx} f(x)\, dx \end{aligned}$$

Thus,

$$M'(0) = \int x e^0 f(x)\, dx = \int x f(x)\, dx = E[X]$$

The 2^{nd} derivative of M(t) is

$$\begin{aligned} M''(t) &= \frac{\partial}{\partial t} M'(t) \\ &= \frac{\partial}{\partial t} E[Xe^{tX}] \\ &= E[\frac{\partial}{\partial t}(Xe^{tX})] \\ &= E[X^2 e^{tX}] \end{aligned}$$

Thus,

$$M''(0) = E[X^2]$$

In general, the n^{th} derivative of M(t) is

$$M^{(n)}(t) = E[X^n e^{tX}] \qquad n \geq 1$$

The n^{th} moment of X is

$$M^{(n)}(0) = E[X^n]$$

1.1 Example 7d Standard Normal Distribution

Let Z be a standard normal random variable with parameters 0 and 1, we have

$$M(t) = E[e^{tZ}] = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz = e^{\frac{t^2}{2}} \qquad \text{(steps in textbook)}$$

$$M'(t) = \frac{2t}{2} e^{\frac{t^2}{2}} = te^{\frac{t^2}{2}} \qquad\qquad M'(0) = 0 = Mean(E[Z])$$

$$M''(t) = e^{\frac{t^2}{2}} + t^2 e^{\frac{t^2}{2}} \qquad\qquad M''(0) = 1 = Var[Z] = E[Z^2]$$

For an arbitrary normal random variable $X = \mu + \sigma Z$ with parameters μ and σ^2,

$$M_X(t) = E[e^{tX}] = E[e^{t(\mu+\sigma Z)}] = e^{t\mu} E[e^{t\sigma Z}] = e^{t\mu} e^{\frac{t^2\sigma^2}{2}} = e^{\mu t + \frac{t^2\sigma^2}{2}}$$

If X, Y are independent, then

$$M(t) = E[e^{t(X+Y)}] = E[e^{tX} e^{tY}] = E[e^{tX}]E[e^{tY}] = M_X(t) M_Y(t)$$

If $M(t) < \infty$ and exists in some region about $t = 0$, then this uniquely defines the probability distribution.

2 Joint Moment Generating Function

If X_1,...,X_n have joint probability density function $f(x_1, ..., x_n)$, then

$$M(t_1, t_2, ..., t_n) = \int ... \int e^{t_1 x_1 + t_2 x_2 + ... + t_n x_n} f(x_1, ..., x_n)\, d\vec{x} = E[e^{t_1 x_1 + t_2 x_2 + ... + t_n x_n}]$$

The individual moment generating functions can be obtained from $M(t_1, ..., t_n)$ by letting all but one of the t_i's be 0. That is,

$$M_{X_i}(t) = M(0, ..., 0, t_i, 0, ..., 0)$$

where the t_i is in the i^{th} place.

3 Additional Properties of Normal Random Variables (Ross 7.8)

Let $Z_1, ..., Z_n$ be a set of n independent standard normal random variables. For some constants a_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$, and μ_i, $1 \leq i \leq m$, define:

$$\begin{aligned} X_1 &= a_{11}Z_1 + ... + a_{1n}Z_n + \mu_1 \\ X_2 &= a_{21}Z_1 + ... + a_{2n}Z_n + \mu_2 \\ &\vdots \\ X_m &= a_{m1}Z_1 + ... + a_{mn}Z_n + \mu_m \end{aligned}$$

The random variables $X_1, ..., X_m$ are said to have a multivariate normal distribution. We can also write the equations in the form of matrix multiplication:

$$\vec{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{pmatrix} = A\vec{Z} + \vec{\mu} = \begin{pmatrix} a_{11} \cdots a_{1n} \\ \vdots \ddots \vdots \\ a_{m1} \cdots a_{mn} \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}$$

$$E[X_i] = \mu_i$$

$$Var[X_i] = Var[\sum_{j=1}^{n} a_{ij}Z_j \ ] = \sum_{j=1}^{n} {a_{ij}}^2$$

$$\begin{aligned} cov(X_i, X_j) &= cov(\sum_{k=1}^{n} a_{ik}Z_k, \sum_{k'=1}^{n} a_{jk'}Z_{k'}) \\ &= \sum_{k,k'} a_{ik'}a_{jk'}cov(Z_k, Z_{k'}) \\ &= \sum_{k=1}^{n} a_{ik}a_{jk} \qquad \text{(inner product of } \vec{a_i} \text{ and } \vec{a_j}) \end{aligned}$$

(Because Z_j's are independent, $cov(Z_k, Z_{k'}) = 0$ unless $k = k'$. So the only non-zero terms in the summation are when $k = k'$)

If $C = AA^T$, then $C = \begin{pmatrix} a_{11} \cdots a_{1n} \\ \vdots \ddots \vdots \\ a_{m1} \cdots a_{mn} \end{pmatrix} \begin{pmatrix} a_{11} \cdots a_{m1} \\ \vdots \ddots \vdots \\ a_{1n} \cdots a_{mn} \end{pmatrix}$, $C_{ij} = \sum_k a_{ik}a_{jk}$ is the covariance of X_i and X_j.

C is the covariance matrix of $\begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix}$:

- C is $m \times m$
- C is symmetric
- C is semi-positive definite

4 The Joint Moment Generating Function for $X_1, ..., X_m$

$$M(t_1, ..., t_m) = e^{\sum_{i=1}^{m} t_i\mu_i + \frac{1}{2}\sum_{i,j} t_it_jC_{ij}}$$

Since M determines the probability density function, and M depends only on μ_i, C_{ij}, then the joint probability density function must only depend on μ_i, C_{ij}.

$$f(x_1, ..., x_m) = \frac{1}{(2\pi)^{\frac{m}{2}}\sqrt{|C|}} e^{-\frac{1}{2}(\vec{X}-\vec{\mu})^T C^{-1}(\vec{X}-\vec{\mu})}$$

$$C = \begin{pmatrix} C_{11} \cdots C_{1m} \\ \vdots \ddots \vdots \\ C_{m1} \cdots C_{mm} \end{pmatrix}$$

4.1 Proposition 8.1

If $X_1, ..., X_n$ are independent and identically distributed normal random variables with mean μ and variance σ^2, then the sample mean $\overline{X}$ and the sample variance S^2 are independent. $\overline{X}$ is a normal random variable with mean μ and variance $\frac{\sigma^2}{n}$; $\frac{(n-1)S^2}{\sigma^2}$ is a chi-squared random variable with $n-1$ degrees of freedom.

5 Additional Examples

5.1 Example 7h

Show that if X and Y are independent normal random variables with respective parameters $(\mu_1, {\sigma_1}^2)$ and $(\mu_2, {\sigma_2}^2)$, then $X+Y$ is normal with mean $\mu_1 + \mu_2$ and variance ${\sigma_1}^2 + {\sigma_2}^2$.

Solution:

$$\begin{aligned} M_{X+Y}(t) &= M_X(t)M_Y(t) \\ &= (e^{\frac{{\sigma_1}^2 t^2}{2} + \mu_1 t})(e^{\frac{{\sigma_2}^2 t^2}{2} + \mu_2 t}) \\ &= e^{(\frac{({\sigma_1}^2 + {\sigma_2}^2)t^2}{2} + (\mu_1+\mu_2)t)} \end{aligned}$$

which is the moment generating function of a normal random variable with mean $\mu_1 + \mu_2$ and variance ${\sigma_1}^2 + {\sigma_2}^2$. The desired result then follows because the moment generating function uniquely determines the distribution.

5.2 Example 7L

Let X and Y be independent normal random variables, each with mean μ and variance σ^2. Show that $X+Y$ and $X-Y$ are independent by computing their joint moment generating function:

Solution:

$$\begin{aligned} E[e^{t(X+Y)+s(X-Y)}] &= E[e^{(t+s)X+(t-s)Y}] \\ &= E[e^{(t+s)X}]E[e^{(t-s)Y}] \\ &= (e^{\frac{\mu(t+s)+\sigma^2(t+s)^2}{2}})(e^{\frac{\mu(t-s)+\sigma^2(t-s)^2}{2}}) \\ &= (e^{2\mu t+\sigma^2 t^2})(e^{\sigma^2 s^2}) \end{aligned}$$

We recognize the preceding as the joint moment generating function of the sum of a normal random variable with mean 2μ and variance $2\sigma^2$ (which is $X+Y$) and an independent normal random variable with mean 0 and variance $2\sigma^2$ (which is $X-Y$). Because the joint moment generating function uniquely determines the joint distribution, it follows that $X+Y$ and $X-Y$ are independent normal random variables.

5.3 Exercise 1

Suppose X has the moment generating function

$$M_X(t) = (1-2t)^{-\frac{1}{2}} \text{ for } t < \frac{1}{2}$$

Find the first and second moments of X.

Solution:

We have

$$\begin{aligned} M_X'(t) &= -\frac{1}{2}(1-2t)^{-\frac{3}{2}}(-2) = (1-2t)^{-\frac{3}{2}} \\ M_X''(t) &= -\frac{3}{2}(1-2t)^{-\frac{5}{2}}(-2) = 3(1-2t)^{-\frac{5}{2}} \end{aligned}$$

So that

$$\begin{aligned} E[X] &= M_X'(0) = (1-2\cdot 0)^{-\frac{3}{2}} = 1 \\ E[X^2] &= M_X''(0) = 3(1-2\cdot 0)^{-\frac{5}{2}} = 3 \end{aligned}$$

5.4 Exercise 2

Suppose that you have a fair 4-sided die, and let X be the random variable representing the value of the number rolled.
(a) Write down the moment generating function of X.
(b) Use this moment generating function to compute the first and second moments of X.

Solution:

(a):

$$\begin{aligned} M_X(t) &= E[e^{tX}] \\ &= e^{1\cdot t}\frac{1}{4} + e^{2\cdot t}\frac{1}{4} + e^{3\cdot t}\frac{1}{4} + e^{4\cdot t}\frac{1}{4} \\ &= \frac{1}{4}(e^{1\cdot t} + e^{2\cdot t} + e^{3\cdot t} + e^{4\cdot t}) \end{aligned}$$

(b): We have

$$\begin{aligned} M_X'(t) &= \frac{1}{4}(e^{1\cdot t} + 2e^{2\cdot t} + 3e^{3\cdot t} + 4e^{4\cdot t}) \\ M_X''(t) &= \frac{1}{4}(e^{1\cdot t} + 4e^{2\cdot t} + 9e^{3\cdot t} + 16e^{4\cdot t}) \end{aligned}$$

so

$$E[X] = M_X'(0) = \frac{1}{4}(1+2+3+4) = \frac{5}{2}$$

and

$$E[X^2] = M_X''(0) = \frac{1}{4}(1+4+9+16) = \frac{15}{2}$$