Nov 10, 2021

Monte Carlo Integration

Suppose we want to compute

$$I = \int f(x)\, p(x)\, dx \qquad \text{where } p(x) \geq 0 \text{ and } \int p(x) = 1$$

Then p defines a valid pdf for some distribution P and therefore $I = \mathbb{E}(f(X))$ where $X \sim P$.

By the law of large numbers, let $X_1, \ldots, X_n$ be IID random variables with distribution P. (and $E(X_i)$, $Var(X_i) < \infty$ for simplicity)

Then

$$I_n = \frac{1}{n}\sum_i f(X_i) \to I \quad \text{as } n \to \infty \quad (\text{ignore form of conv.})$$

And

$$\mathbb{E}(I_n) = I$$

$$Var(I_n) = \frac{1}{n}\mathbb{E}\left((f(X) - I)^2\right)$$

$$\approx \frac{1}{n}\underbrace{\left(\frac{1}{n-1}\sum_{i=1}^{n}\left(f(x_i) - \frac{1}{n}\sum_{1}^{n} f(x_i)\right)^2\right)}_{\text{sample variance } \hat{\sigma}^2}$$

x_i is sample

X_i is random variable

Note: Can transform to:

$$\int f(x)\, dx = \int \frac{f(x)}{p(x)}\, p(x)\, dx$$

The question remains: how do we draw from P?

Computational Statistics

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Random Number Generation

Using everything we've learned so far, how can we generate random variables from arbitrary distributions? (In one-dimension for now.)

The basic building block is a method to generate $U(0,1)$ random numbers – languages have built in methods for this. (Be sure always to "set the seed" to make your code reproducible.)

Goal: Find f such that $X = f(U)$ has desired distribution.

Inverse CDF Method

Trivially, if X has pdf φ and CDF Φ, then $\Phi(X) \sim U$, i.e., draw x from X, compute $u = \Phi(x)$, u is then a draw from $U(0,1)$:

$$\mathbb{P}(\Phi(x) \leq u) = \mathbb{P}(X \leq \Phi^{-1}(u))$$

↑ cont., monotone

$$= \Phi(\Phi^{-1}(u)) = u$$

$$\Rightarrow \Phi(x) \sim U(0,1).$$

Alternatively:

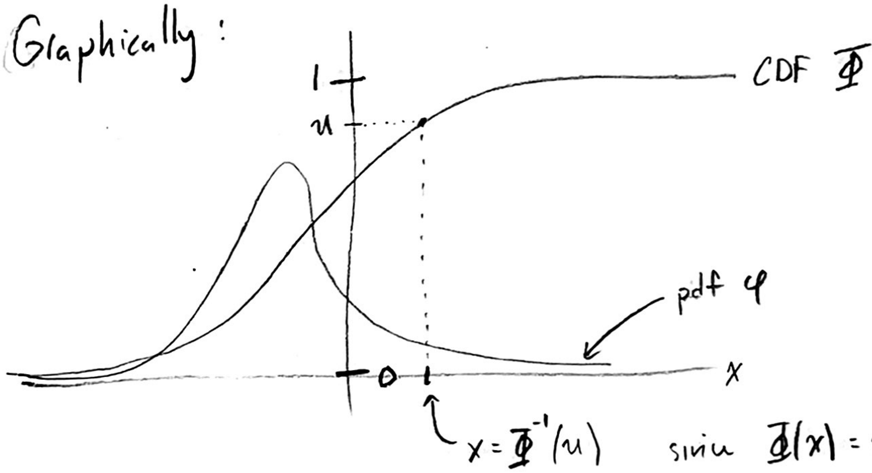
Draw $u \sim U(0,1)$, then compute $x = \Phi^{-1}(u)$.

$$\Rightarrow \quad \mathbb{P}(X \le z) = \mathbb{P}(\Phi^{-1}(U) \le z)$$

$$= \mathbb{P}(U \le \Phi(z)) = \Phi(z)$$

$\Rightarrow$ X has CDF Φ!

Graphically:



since $\Phi(x) = u$.

Computationally:

$$\Phi(x) = \int_{-\infty}^{x} f(y)\, dy \quad \Rightarrow \quad \text{solve} \underbrace{\int_{-\infty}^{x} f(y)\, dy}_{\Phi(x)} = u$$

u ← drawn between (0,1)

Newton's Method says:

$$x_{k+1} = x_k - \frac{\Phi(x_k) - u}{\Phi'(x_k)} = x_k - \frac{\Phi(x_k) - u}{\varphi(x_k)}$$

$\Phi(x_k)$ ← can often be expensive to compute

$\varphi(x_k)$ ← can be small

Note:

let $s_k := x_{k+1} - x_k$

then $\displaystyle\underbrace{\int_{-\infty}^{x_{k+1}} f(y)\,dy}_{\Phi(x_{k+1})} = \int_{-\infty}^{x_k} f(y)\,dy + \int_{x_k}^{x_{k+1}} f(y)\,dy$

$$\Phi(x_{k+1}) = \Phi(x_k) + \mathbb{P}(X \in (x_k, x_{k+1}))$$

← alternative way to compute $\Phi(x_{k+1})$ and reuse $\Phi(x_k)$.