The Discrete Fourier Transform

The continuous case: any smooth, periodic, complex-valued function $f : [0, 2\pi) \to \mathbb{C}$ can be written in terms of its Fourier series:

$$f(\theta) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta}$$

$$\alpha_n = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} \, d\theta \qquad \text{(by the orthogonality of } e^{in\theta}, \dots)$$

The "smoother" f is, the faster the coefficients α_n decay.

Let's imagine a discretization of the formula for α_n using the trapezoidal rule:

$$\alpha_n \approx \frac{1}{2\pi} \frac{1}{N} \sum_{l=1}^{N} f(\theta_l) e^{-in\theta_l}, \qquad \theta_l = \frac{(l-1)2\pi}{N}$$

$$= \frac{1}{2\pi} \frac{1}{N} \sum_{l=1}^{N} f(\theta_l) e^{-in(l-1)2\pi/N}$$

With this in mind, we now <u>define</u> the Discrete Fourier Transform as:

$$\hat{f}_k = \sum_{l=1}^{N} f_l \, e^{-2\pi i (k-1)(l-1)/N} \, . \qquad k = 1, \dots, N.$$

One interpretation of is as ~~an~~ computing an approximation to α_k, the <u>true</u> Fourier series coefficient of f. (properly scaled)

(f_j are merely values, they can be assumed to be equispaced samples of a function on some interval).

$\hat{f}_k \approx \alpha_k$ <u>if</u> $N \gtrsim 2k$. (Nyquist sampling for periodic functions).

The Inverse DFT is:

$$f_j = \sum_{k=1}^{N} \hat{f}_k \, e^{2\pi i (k-1)(j-1)/N}$$

Let F = DFT matrix: $F_{jk} = e^{-2\pi i (j-1)(k-1)/N}$.

$F^* F = N \cdot I$ (I.e. $F^{-1} = \frac{1}{N} F^*$)

complex conjugate transpose.

F^* = IDFT matrix

$$(F^*F)_{jk} = \sum_{n=1}^{N} e^{2\pi i (j-1)(n-1)/N} e^{-2\pi i (n-1)(k-1)/N}$$

$$= \sum_{n=1}^{N} e^{\frac{2\pi i (n-1)(j-1-k+1)}{N}}$$

$$= \sum_{n=1}^{N} e^{2\pi i (n-1)(j-k)/N}$$

If $j=k$, $\Rightarrow \sum e^0 = N$

If $j \neq k$, $\Rightarrow \sum e^{2\pi i (n-1)(j-k)/N} = e^{-2\pi i (j-k)/N} \underbrace{\sum_{n=1}^{N} e^{2\pi i n/N}}_{=0.}$

HW exercise.

The direct application of F or F^* requires $\mathcal{O}(n^2)$ calculation since they are dense.

Alternative forms:

$$\hat{f}_k = \sum_{\ell=-\frac{N}{2}+1}^{N/2} f_\ell e^{-2\pi i \ell k/N}$$ (N even, slightly more convenient)

or $$\hat{f}_k = \sum_{\ell=-\frac{(N-1)}{2}}^{\frac{N-1}{2}} f_\ell e^{-2\pi i \ell k/N}$$ (N odd).

For simplicity, let's look at the form

$$\hat{f}_k = \sum_{l=0}^{N-1} f_l \, e^{-2\pi i k l / N}, \quad k = 0, \dots, N-1$$

from now on.

Can we compute this sum, for all k, faster than $\mathcal{O}(N^2)$? Observe that:

$$\hat{f}_k = \sum_{l=0}^{N-1} f_l \, e^{-2\pi i k l / N} = \sum_{l \text{ even}} f_l \, w_N^{-kl} + \sum_{l \text{ odd}} f_l \, w^{-kl}$$

when $w_N = e^{2\pi i / N}$

$$= \sum_{l=0}^{N/2} f_{2l} \, w_N^{-k(2l)} + \sum_{l=0}^{N/2} f_{2l+1} \, w_N^{-k(2l+1)}$$

$$= \sum_{l=0}^{N/2} f_{2l} \underbrace{w_{N/2}^{-kl}}_{e^{-2\pi i k l / N/2}} + w_N^{-k} \sum_{l=0}^{N/2} f_{2l+1} \underbrace{w_{N/2}^{-kl}}_{e^{-2\pi i k l / N/2}}$$

This is the sum of <u>two</u> DFTs, each of size $N/2$.

But $k = 0, \ldots, N-1$

what happens for $k > \frac{N}{2} - 1$

$$\omega_{N/2}^{-kl} = e^{-2\pi i k l / N/2}$$

Let $k = \frac{N}{2} + j$, $\quad j \geq 0, \ldots$

$$= e^{-2\pi i (N/2 + j) \cdot l / N/2}$$

$$= \underbrace{e^{-2\pi i l}}_{=1} \cdot e^{-2\pi i j l / N/2} = e^{-2\pi i j l / N/2} =$$

We say that $k = \frac{N}{2} + j$ <u>aliases</u> to the ~~fre~~ frequency j, (or mode j).

So
$$F_N \vec{f} = \begin{pmatrix} F_{N/2} & W_N F_{N/2} \\ F_{N/2} & -W_N F_{N/2} \end{pmatrix} \begin{pmatrix} \vec{f}_{even} \\ \vec{f}_{odd} \end{pmatrix}$$

$$W_N = \begin{pmatrix} w_N^0 & & \\ & \ddots & \\ & & w_N^{N/2-1} \end{pmatrix}$$

since $e^{-2\pi i (\frac{N}{2} + j)/N} = -e^{-2\pi i j / N}$

$$= \begin{pmatrix} F_{N/2} & \boxed{W_N} F_{N/2} \\ F_{N/2} & \boxed{-W_N} F_{N/2} \end{pmatrix} \underbrace{\begin{pmatrix} 1\,0\,0\,0 \cdots \\ 0\,0\,1\,0 \cdots \\ 0\,0\,0\,0\,1 \\ \cdots \\ 0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0 \cdots \end{pmatrix}}_{\text{extract even and odd parts}} \hat{f}$$

diagonal matrix

The cost for this calculation is

$$\underbrace{2\cdot \mathcal{O}\left(\left(\tfrac{N}{2}\right)^2\right)}_{\text{the DFTS}} + \underbrace{2\cdot \tfrac{N}{2}}_{\text{Scaling by } W_N \ (\omega_N^{-k})} + \underbrace{2\cdot N}_{\text{Addition}}$$

$$= \underbrace{\mathcal{O}\left(\tfrac{N^2}{2}\right)}_{\text{decreased by factor of two.}} + \mathcal{O}(N)$$

Continuing to split each of the $F_{N/2}$'s we eventually arrive at a cost of $\mathcal{O}(N \underbrace{\log_2 N}_{})$

there are $\log_2$ splittings that occur.

(In fact, the cost is basically $= \underbrace{5N\log_2 N}_{\text{Very fast.}}$.)

<u>Note</u> This algorithm was <u>exact</u> ← consequence of the algebraic properties of $e^{-2\pi i jk/N}$. The <u>only</u> source of error in the computation is numerical round-off, but $K(F) = 1$, so this is very minimal.

Computing convolution with the FFT.

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In the continuous case, $f * g$ is given by

$$(f*g)(x) = \int f(x-t)\, g(t)\, dt$$

$$= \int f(t)\, g(x-t)\, dt$$

the Fourier transform of a convolution is given by the product of the Fourier transforms:

$$\mathcal{F}(f*g) = \hat{f} \cdot \hat{g}$$

Discretely, the N-point cyclic convolution is given by

$$h = f * g$$

$$h_n = \sum_{l=0}^{N-1} f_{n-l}\, g_l = \sum_{l=0}^{N-1} f_l\, g_{n-l}$$

where f, g are periodic sequences.

The DFT of $\mathcal{D}(f*g) = \mathcal{D}(f) \cdot \mathcal{D}(g)$

i.e. $\hat{h}_n = \hat{f}_n \cdot \hat{g}_n$.

But the point is that this <u>actual</u> convolution can be computed fast using the FFT.

since $h = \frac{1}{N} \mathcal{D}^{-1} \mathcal{D}(f * g)$

$$= \frac{1}{N} \mathcal{D}^{-1}(\hat{f} \cdot \hat{g})$$

compute using FFT, $O(N \lg N)$

$$= f * g \text{ by IFFT, } O(N \log N).$$

<u>Exact</u> statement, again.

Fun things with the FFT

Clenshaw-Curtis quadrature: Expand function f in Chebyshev series ↑ cosine series and then integrate. Direct cost: $\mathcal{O}(n^2)$ if f is sampled at n points.

→ Accelerate via the FFT.

$$\int_{-1}^{1} f(x)\,dx = \int_0^\pi f(\cos\theta)\sin\theta\,d\theta \qquad (\text{let } x = \cos\theta)$$

If $f(\cos\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\theta$, then we can do this integral exactly:

$$\int_0^\pi \sin\theta\,d\theta = 2$$

$$\int_0^\pi \cos k\theta \sin\theta\,d\theta = \begin{cases} 0 & k \text{ odd} \\ \frac{2}{1-k^2} & k \text{ even} \end{cases}$$

$$\Rightarrow \int_0^\pi f(\cos\theta)\sin\theta\,d\theta = \int_0^\pi \sin\theta\left(\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos k\theta\right)d\theta$$

$$= a_0 + \underbrace{\sum_{k=1}^{\infty} \frac{2a_{2k}}{1-4k^2}}_{\text{only even terms.}}$$

This computation requires that we compute the coefficients $a_0, a_1, \ldots$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\cos\theta)\ d\theta$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\cos\theta)\cos n\theta\ d\theta$$

How can we compute these terms using the FFT?

<u>One option</u>:

$$f(\cos\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta$$ is the <u>cosine</u> series for $f(\cos\theta)$

Alternatively, f <u>also</u> admits a standard Fourier series:

$$f(\cos\theta) = \sum_{-\infty}^{\infty} \beta_n e^{in\theta}$$

Equating terms, we see that

$$\beta_0 = \frac{a_0}{2}$$

$$\cancel{\beta_n e^{in\theta} + \beta_{-n} e^{-in\theta} = \beta_n \cos n\theta + i\beta_n \sin n\theta}$$

$$\cancel{+ \beta_{-n}\cos n\theta - i\beta_{-n}\sin n\theta}$$

$$\cancel{= \underbrace{(\beta_n+\beta_{-n})}_{=a_n}\cos n\theta + i\underbrace{(\beta_n-\beta_{-n})}_{=0}\sin n\theta}$$

Recall, $\cos n\theta = \dfrac{e^{in\theta} + e^{-in\theta}}{2}$

$$\Rightarrow f(\cos\theta) = \frac{a_0}{2} + \sum_{n \neq 0} \frac{a_n}{2} e^{in\theta}$$

$$\Rightarrow \mathrm{Re}(\beta_n) = \frac{a_n}{2}$$

We only require that $\mathrm{Im}(\beta_n) = -\mathrm{Im}(\beta_{-n})$

Since if $\varphi(\theta)$ is <u>real</u>, then

$$\varphi(\theta) = \sum_{-\infty}^{\infty} \alpha_n e^{in\theta}$$

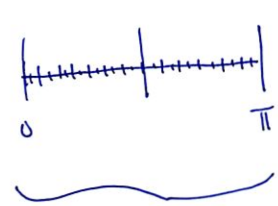
$$\overline{\varphi(\theta)} = \overline{\sum_{-\infty}^{\infty} \alpha_n e^{in\theta}} = \sum_{-\infty}^{\infty} \overline{\alpha_n}\, e^{-in\theta} = \sum_{-\infty}^{\infty} \overline{\alpha_{-n}}\, e^{in\theta}$$

$$\Rightarrow \alpha_n = \overline{\alpha_{-n}} \checkmark$$

Therefore by computing the FFT of $f(\cos\theta)$ to calculate β_n's, we can compute these integrals.

<u>Advantages</u>: Nested quadrature:

$$\int_0^{\pi} f = \int_0^{\pi/2} f + \int_{\pi/2}^{\pi} f$$



splitting the interval does <u>not</u> require the eval of f at <u>all</u> new points.

Option 2 for computing the coefficients a_n:

The discrete cosine transform: (DCT)

Very similar to the DFT, but the identities are slightly more complicated.

$$\hat{x}_k = \sum_{l=0}^{N-1} x_l \cos\left(\frac{\pi k}{N}\left(l+\frac{1}{2}\right)\right) \quad \text{for } k = 0, \dots, N-1$$

other forms:

$$= \underbrace{x_0 + (-1)^k x_{N-1} + 2\sum_{l=1}^{N-2} x_l \cos\left(\frac{\pi k l}{N-1}\right)}_{\text{clearly the trapezoidal rule on } [0,\pi]}$$

Connection with Chebyshev polynomials

$$f(x) = \sum_{n=0}^{\infty} \alpha_n T_n(x) \qquad \text{under} \quad x = \cos\theta$$

$$= \sum_{n=0}^{\infty} \alpha_n \cos(n \operatorname{acos} x)$$

$$f(\cos\theta) = \sum \alpha_n \cos(n\theta)$$ ← FFT techniques can be used to compute a_n.