Homework 9

Due: 2:00pm April 14, 2016

Each problem is worth 10 points.

Exercise 1 : Consider the initial value problem $y' = \lambda y$, $y(0) = 1$. Show that the local truncation error of the classical fourth-order Runge-Kutta method is $\mathcal{O}(h^4)$:

$$y_{k+1} = \left(1 + h\lambda + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{6} + \frac{h^4\lambda^4}{24}\right) y_k$$

Exercise 2 : Show that the classical fourth-order Runge-Kutta method is stable by showing that when it is written in the form

$$y_{k+1} = y_k + h\Psi(t_k, y_k, h)$$

the function Ψ satisfies a Lipschitz condition.

Exercise 3 : Show that the local truncation error in the multi-step method

$$y_{k+2} - 3y_{k+1} + 2y_k = h\left(\frac{13}{12} f(t_{k+2}, y_{k+2}) - \frac{5}{3} f(t_{k+1}, y_{k+1}) - \frac{5}{12} f(t_k, y_k)\right)$$

is $\mathcal{O}(h^2)$.

Exercise 4: Let the $n \times n$ matrix $\mathbf{A}$ be defined as:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -\alpha_0 & \cdots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix}.$$

Show that the characteristic polynomial $\rho(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I})$ of the matrix $\mathbf{A}$ is

$$\rho(\lambda) = (-1)^n \left(\lambda^n + \sum_{\ell=0}^{n-1} \alpha_\ell \, \lambda^\ell\right).$$

Hint: Expand the determinant using the last row.