Lecture 1 1/23/18 Numerical Analysis

Logistics

~~Course topics~~

- Office hours Wed 2:30-4:30
- email me ~~[illegible]~~

- Use piazza as communication tool

- Recitation

  Fri: 1230-145 CIWW 101

- webpage linked from cims.nyu.edu/~oneil

Topics

- solving nonlinear systems of equations
- Numerical linear Algebra
- Polynomial Interpolation
- Numerical Integration
- ODEs (Initial value problems)
- Time permitting: Monte Carlo Methods, Fast Fourier Transform.

Computing

There is an element of computing / programming for this course. (Beware!)

- Hands on approach to numerical analysis is the best way to learn.

- In class Matlab examples when appropriate

  - Recitation will provide intro to MATLAB during first two weeks.

  - Matlab accessible via Courant license / labs.

  - Other languages are fine (Python, C) ~~but plotting data may be required~~

  - Nice to make plots of data when programming

Textbook

- Süli & Mayers, Intro to Numerical Analysis (PDF via NYU)

- other references suggested when appropriate

Grading

Homework (30%) (~bi-weekly)

Quizzes in Recitation (10%) (~biweekly)

Midterm (25%) (March 8)

Final (35%) (TBD)

- HW is to be turned in IN CLASS
  - Mixed paper/programming
  - Collaborate is fine, but write up *your own* solution, and write *your own* code
  - See NYU Academic Integrity

Overview

- Numerical Analysis at Courant
- Numerical Math failures.

(4)

Development of Numerical Methods at Courant
A few examples...

- Fast multipole method (FMM) (Greengard, O'Neil, Zorin,...)
- Immersed boundary method for solid-fluid interactions (Peskin)
- Adaptive mesh and cut cell methods for hyperbolic equations (Berger)
- Methods for studying dynamical systems, multiscale methods (Vanden-Eijnden)
- Methods for free boundary problems in fluid dynamics (Shelley)
- Scalable implicit solvers for viscous flows (Donev, Stadler)
- Sampling methods and Uncertainty Quantification (Goodman, Stadler)
- ...

Applications of Numerical Methods at Courant
A few examples...

- Numerical simulation of Tsunami waves and flooding (Berger)
- Simulation and analysis of natural and artificial heart valves (Peskin)
- Simulation of plate tectonics and mantle convection (Stadler)
- The physics of cell's interiors and their motion (Shelley, Mogilner)
- Comutational fluid/hydrodynamics (Donev)
- Optimal complexity wave simulations (Greengard)
- Simulation of blood cells-resolving blood flow (Zorin)
- ...

Famous numerical mathematics failures
Patriot Missile Failure

In the 1991 Golf War, a patriot missle failed to intercept an Iraqi Scud missile.

28 US soulders died, 100 were injured.

Cause: Inaccurate calculation of the time since boot due to computer arithmetic errors

http://www.ima.umn.edu/~arnold/disasters/patriot.html

Famous numerical mathematics failures
Sinking of Sleipner oil platform

An oil platform in the North Sea sank near Stavanger (Norway) in 1991. Top part weights 57,000 tons, supposed to support drilling equipment that weights 40,000 tons.

Total economic loss was about 700 million USD.

Cause: Weak parts in the base could not resist the weight. Stresses were underestimated by 47%, leading to insufficient design. This was mainly due to an inaccurate finite element calculation to solve the PDE.

http://www.ima.umn.edu/~arnold/disasters/sleipner.html

Famous numerical mathematics failures
Explosion of Ariane 5

Unmanned Ariane 5 rocket launched by the European space agency expolded in 1996.

Rocket value was abot 500 million USD.

Cause: Conversion of a floating point number to an integer led to "overflow" resulting in complete loss of guidance and altitude information 37 seconds after start.

http://www.ima.umn.edu/~arnold/disasters/ariane.html

First Topic   Solving non-linear equations:

Linear: $3x + 7 = 2$

$$\Rightarrow x = \frac{2-7}{3}$$

Non-linear $\cos(x) + x^2 - 7 = 5$

Harder to solve for x...

Recall: Linear Transformation from Lin. Alg.

$$T(x+y) = T(x) + T(y)$$

$$T(ax) = a \cdot T(x).$$

Ex: Quadratic Formula

$$ax^2 + bx + c = 0$$

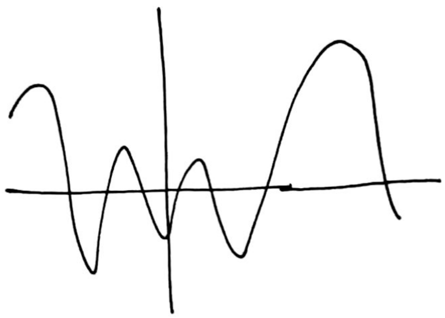
$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

No such formula exists for deg ≥ 5..!

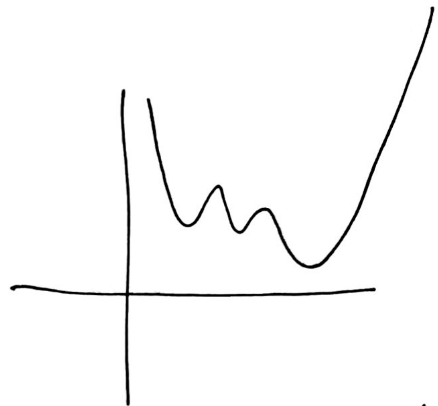
Question? Does $f(x) = 0$ have a solution (in general)?

$\uparrow$ non-linear

Graphically:



vs.

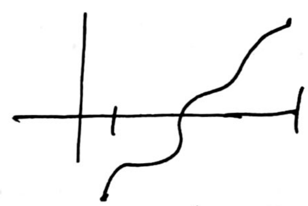


many solutions

no solution (at least if x is required to be real)

Ex: $x^2+1=0 \Rightarrow x = \pm i$
$= \pm\sqrt{-1}$

Obvious criteria for solution to exist in $[a,b]$: $f(a) < 0$ and $f(b) > 0$, f continuous:

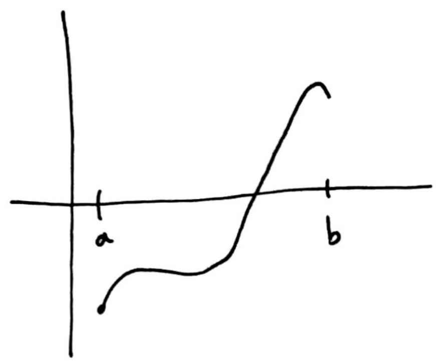


Thm: Let f be real-valued, continuous on $[a,b]$, and $f(a)\cdot f(b) < 0$. Then, there exist $x \in (a,b)$ s.t. $f(x) = 0$.

Proof: Simply by Intermediate Value Theorem (Calc I).

Can we use this to develop a numerical algorithm?

Bisection



$f(a) < 0$

$f(b) > 0$

examine the midpoint: $\frac{a+b}{2}$

- If $f\left(\frac{a+b}{2}\right) \le 0$, then

  $x \in \left[\frac{a+b}{2}, b\right]$.

- If $f\left(\frac{a+b}{2}\right) \ge 0$, then $x \in \left(a, \frac{a+b}{2}\right]$

Put another way, compute $p_a = f(a)\, f\left(\frac{a+b}{2}\right)$

$p_b = f\left(\frac{a+b}{2}\right) f(b)$

If $p_a \le 0$, $x \in \left(a, \frac{a+b}{2}\right] \rightarrow$ set $\left\{\begin{matrix} a = a \\ b = \frac{a+b}{2} \end{matrix}\right\}$, repeat

If $p_b \le 0$, $x \in \left[\frac{a+b}{2}, b\right)$. $\rightarrow$ set $\left\{\begin{matrix} a = \frac{a+b}{2} \\ b = b \end{matrix}\right\}$, repeat.

<u>If</u> $f(a)\, f(b) > 0$, we <u>cannot</u> draw any conclusions.

This algorithm is called <u>bisection</u>

How much does the error go down on every iteration?

$x_k \in (a_k, b_k)$

$x_{k+1} \in (a_{k+1}, b_{k+1})$ but $\frac{|b_{k+1} - a_{k+1}|}{|b_k - a_k|} = \frac{1}{2}$

$$\Rightarrow |b_k - a_k| = \frac{|b_0 - a_0|}{2^k}$$

The error $|x - x_k^*|$ goes down by a factor of 2 every iteration.

If we require that $|x - x_k| < \epsilon$ ← error tolerance,

then we need $\frac{|b_0 - a_0|}{2^k} < \epsilon$

$$\Rightarrow k \geq -\log_2 \frac{\epsilon}{|b_0 - a_0|}$$

This used only information of function values, and even then, only the sign of them. Can we do better?

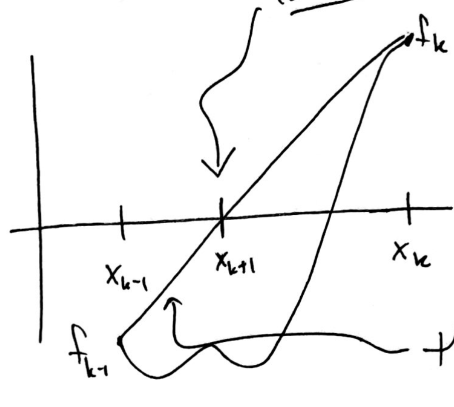
Imagine: $f(a) = -.000001$. $f(b) = 10$.

Is it more likely for the root to be near a or b? a of course!

Next method: The secant method

Graphically

solve $f(x) = 0$



$$s(x) = \frac{f_k - f_{k-1}}{x_k - x_{k-1}}(x - x_k) + f_k$$

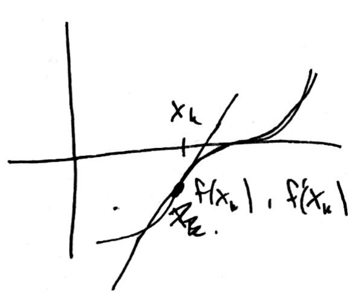
Its root (zero) is given by $x_{k+1} = x_k - f_k\left(\frac{x_k - x_{k-1}}{f_k - f_{k-1}}\right)$

Repeat this procedure with x_k, x_{k+1}.

We will revisit the rate of convergence of this method later...

Ok: So far, we've only used function values, how about derivative information? $f'(x_k)$?

If we only knew one value and its derivative, what would we do?



The equation of the tangent line is given by:

$$t(x) = f(x_k) + f'(x_k)(x - x_k)$$