Lecture 4 Numerical Analysis 2/1/18

Last time:

- Fixed point iterations
- (keep in mind that all our root finding algorithms are just fixed point methods. Ex: Newton:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$\iff \quad x = x + \frac{f(x)}{f'(x)}$$

$\implies$ trivially true if $f(x)=0$.

- Statement of the Contraction Mapping Thm.

Recall:

Definition (Contraction) Let g be continuous on $[a,b]$. The function g is a contraction on $[a,b]$ if there exists a number L with $0 < L < 1$ such that

$$\underbrace{|g(x) - g(y)| < L|x-y|}_{\text{This means that } g \text{ maps points to values which are closer together.}} \quad \text{for all } x,y \in [a,b].$$

If L is allowed to be any positive number, this is known as a Lipschitz condition.

<u>Thm</u> Contraction Mapping Theorem

Let g be continuous on $[a,b]$, $g(x) \in [a,b]$, and g is a contraction on $[a,b]$.

Then g has a unique fixed point $\xi = g(\xi)$. Furthermore, $x_{k+1} = g(x_k)$ converges to ξ for <u>any</u> $x_0 \in [a,b]$.

<u>Proof</u>: A fixed point ξ exists due to Brouwer's Fixed Point Thm. Now, suppose that ξ is <u>not</u> unique. Then there exists another fixed point, ξ', such that $|\xi - \xi'| = |g(\xi) - g(\xi')| \leq L|\xi - \xi'|$.

So $(1-L)|\xi - \xi'| \geq 0 \Rightarrow \underline{(1-L) \geq 0}$.

Contradiction.

Now, we will show that $\{x_k\}$ converges for any initial $x_0 \in [a,b]$.

Since g is a contraction,

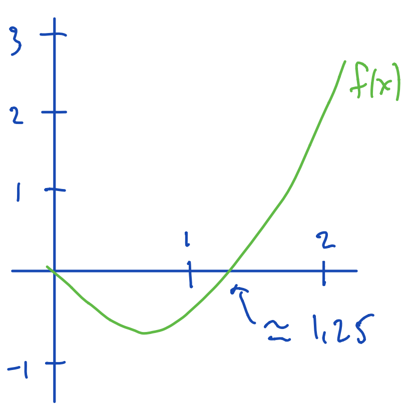
$$|x_k - \xi| = |g(x_{k-1}) - g(\xi)| \le L\,|x_{k-1} - \xi|$$

$$\Rightarrow \quad |x_k - \xi| \le L^k\,|x_0 - \xi|$$

and since $0 < L < 1$, $\lim_{k\to\infty} L^k = 0$,

and therefore $\lim_{k\to\infty} |x_k - \xi| = 0$.

<u>Ex:</u> Look at $f(x) = e^x - 2x - 1$ on $[1,2]$. $f(x) = 0$ has a unique solution, ξ. This



can be re-written as a fixed point problem:

$\Rightarrow \quad e^x - 2x - 1 = 0$

$\Rightarrow \quad e^x = 2x + 1$

$\log e^x = \log(2x+1)$

$$x = \underbrace{\log(2x+1)}_{\text{denote this function } g.}$$

- g is continuous on $[1,2]$, and differentiable.

By the MVT,

$$|g(x) - g(y)| = |g'(\eta)(x-y)| \quad \text{some } \eta \in [1,2]$$

$$= |g'(\eta)|\,|x-y|$$

$$\hookrightarrow = \frac{1}{2x+1} \in \left[\tfrac{2}{5}, \tfrac{2}{3}\right]$$

$$\leq \frac{2}{3}|x-y|$$

So g is a contraction.

Therefore, by the Contraction Mapping Theorem, $x_{k+1} = g(x_k)$ converges for any $x_0 \in [1,2]$ to ξ, the root of f.

How many iterations do we need to do to guarantee that $|x_k - \xi| \le \epsilon$?

Well, from CMT proof,

$$|x_k - \xi| \le L^k |x_0 - \xi|$$

$$\le L^k |b-a|$$

trivially.

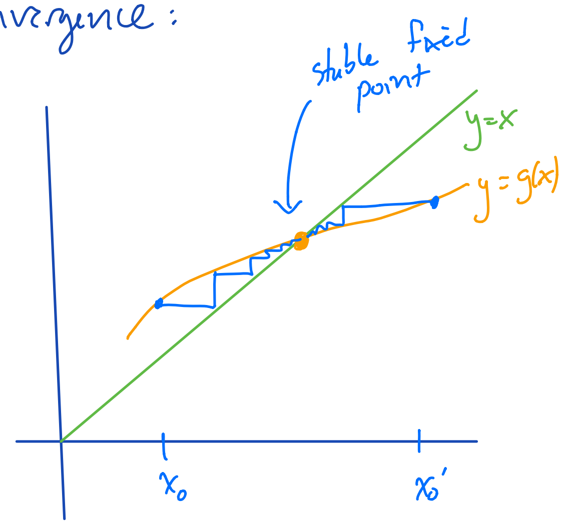
If $L^k |b-a| \le \epsilon$, then

$$k \log L \le \log \frac{\epsilon}{|b-a|}$$

$$\Rightarrow k \ge \frac{1}{\log L} \log \frac{\epsilon}{|b-a|}$$

note since $\log L$ is negative.

Now a few notes on convergence:

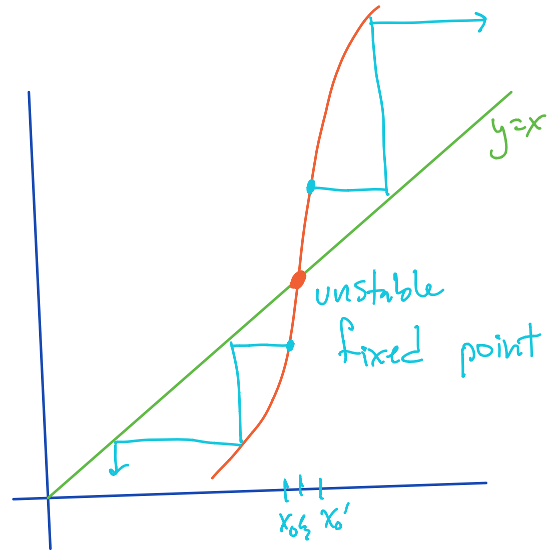


Stable fixed point

ξ is stable if $x_k \to \xi$ for every x_0 in some sufficiently small neighborhood of ξ.

Unstable fixed point

ξ is unstable if the only initial condition that yields a convergent sequence is $x_0 = \xi$ (i.e. x_k diverges for every x_0 in a neighborhood of ξ)



If ξ is a stable fixed point, at what rate does the sequence converge?

$\Rightarrow$ Look at the ratio of successive errors:

$$\lim_{k\to\infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|} = \lim_{k\to\infty} \frac{|g(x_k) - \xi|}{|x_k - \xi|} = |g'(\xi)|$$

So the derivative at ξ dictates how fast the sequence converges.

Note: $|g'(\xi)| < 1$, otherwise $|x_{k+1} - \xi| > |x_k - \xi|$ and the seq. diverges.

Definitions of rates:

Let the error be

$$|x_k - \xi| = \epsilon_k,$$

and set $\displaystyle\lim_{k\to\infty} \frac{\epsilon_{k+1}}{\epsilon_k} = \mu.$

If $\mu \in (0,1)$ then $\{x_k\}$ convergs linearly.

Set $\rho = -\log_{10}\mu.$

ρ is the asymptotic rate of convergence.

Example: $x_k = 1 + \frac{1}{10^k}$

$$\lim_{k\to\infty} \frac{\left|1 + \frac{1}{10^{k+1}} - 1\right|}{\left|1 + \frac{1}{10^k} - 1\right|} = \lim_{k\to\infty} \frac{1}{10^{k+1}} \Big/ \frac{1}{10^k} = \frac{1}{10}$$

$$\rho = -\log_{10}\frac{1}{10} = \log_{10} 10 = 1$$

What does ρ measure: the number of correct decimal digits gained in one iteration.

For fixed point iterations, since

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k} = |g'(\xi)| \ , \quad \mu = |g'(\xi)|$$

and $\rho = -\log_{10} |g'(\xi)|$

This means that the flatter the function g is near the fixed point, the faster $x_{k+1} = g(x_k)$ converges.

Example