Lecture 15 Numerical Analysis March 26, 2018

Last time:

- Eigenvalue review
$$A\underline{v} = \lambda \underline{v}$$
$$\Leftrightarrow \det(A - \lambda I) = 0$$

- Gerschgorin's Theorem

All λ lie in $\bigcup_i D_i$ with
$$D_i := \{ z \in \mathbb{C} \text{ s.t. } |z - a_{ii}| \le \sum_{j \ne i} |a_{ij}| \}$$

- The Power Method

If $|\lambda_j| > |\lambda_\ell|$ for all $\ell \ne j$, then
$$A^k \underline{r} = c_1 \lambda_1^k \hat{\underline{v}}_1 + \dots + c_n \lambda_n^k \hat{\underline{v}}_n$$
$$\approx c_j \lambda_j^k \hat{\underline{v}}_j$$

randomly chosen vector

($\hat{\underline{x}}$ denotes a unit vector)

This means that

$$\frac{A^{k+1}\underline{r}}{\|A^k \underline{r}\|} \approx \frac{c_j \lambda_j^{k+1} \hat{\underline{v}}_j}{|c_j|\,|\lambda_j^k|\,\|\underline{v}_j\|} \approx \lambda_j \hat{\underline{v}}_j$$

unit vector, j^{th} eigenvector.

The Power Method

Start with random $\underline{w}_0$.

1. compute $\hat{\underline{w}}_0 = \dfrac{\underline{w}_0}{\|\underline{w}_0\|}$ ← 2-norm.

2. set $\underline{w}_{k+1} = A\hat{\underline{w}}_k$.

3. (normalize $\hat{\underline{w}}_{k+1} = \dfrac{\underline{w}_{k+1}}{\|\underline{w}_{k+1}\|}$)

To calculate (approximate) λ_1, when $|\lambda_1| > |\lambda_\ell|$, $\ell \neq 1$, then are two basic options:

1. $\lambda_1 \approx (A\hat{\underline{w}}_k)_i / (\hat{\underline{w}}_k)_i$ — this may be small; ith component

2. set $\lambda_1 \approx (A\hat{\underline{w}}_k, \hat{\underline{w}}_k)$ (since $\|\hat{\underline{w}}_k\| = 1$)

How fast does the power method converge?

Examine the quantity $\hat{\underline{w}}_k - \hat{\underline{v}}_1$:

If k is sufficiently large, then

$$\underline{w}_k \approx c_1 \lambda_1^k \hat{\underline{v}}_1 \, , \quad \text{then} \quad (\text{assume } c_1 > 0)$$

$$\hat{\underline{w}}_k \approx \frac{1}{|c_1| |\lambda_1^k|} \left(c_1 \lambda_1^k \hat{\underline{v}}_1 + c_2 \lambda_2^k \hat{\underline{v}}_2 + \dots + c_n \lambda_n^k \hat{\underline{v}}_n \right)$$

$$= \hat{\underline{v}}_1 + \frac{c_2}{c_1} \frac{\lambda_2^k}{|\lambda_1^k} \hat{\underline{v}}_2 + \dots$$

Therefor

$$\| \hat{\underline{w}}_k - \hat{\underline{v}} \| \approx \| \hat{\underline{v}}_1 + \frac{c_2}{c_1} \frac{\lambda_2^k}{|\lambda_1^k|} \hat{\underline{v}}_2 + \dots - \hat{\underline{v}}_1 \|$$

$$\sim \mathcal{O}\left(\left| \frac{\lambda_2}{\lambda_1} \right|^k \right)$$

the convergence of the Power Method depends on the gap in the eigenvalues.

This means that if $\left|\frac{\lambda_2}{\lambda_1}\right| \approx 1$ then convergence is <u>very</u> slow.

How can this idea be accelerated?

<u>Idea one:</u> Power method with shifts:

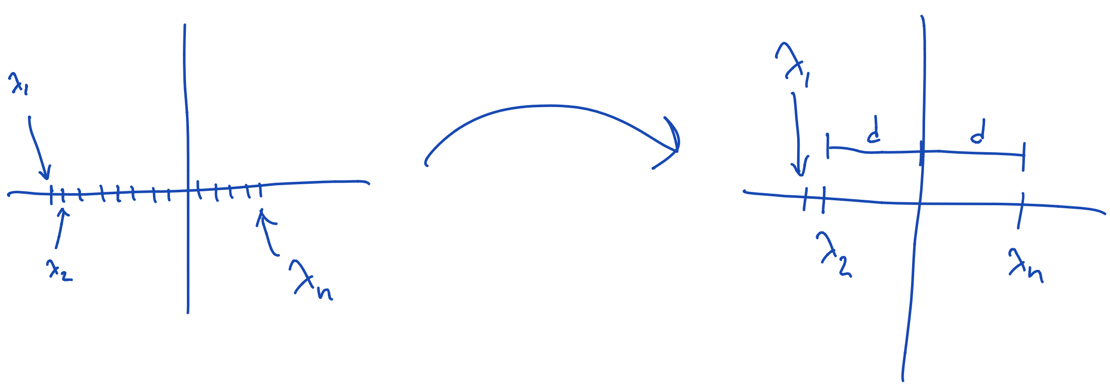
If A has eigenvalues $\lambda_1, \dots, \lambda_n$,

then $A - \lambda \underline{I}$ has eigenvalues $\lambda_1 - \lambda, \dots, \lambda_n - \lambda$.

<u>Pf:</u>

$$\begin{aligned}(A - \lambda \underline{I})\underline{v}_\ell &= A\underline{v}_\ell - \lambda \underline{v}_\ell \\ &= \lambda_\ell \underline{v}_\ell - \lambda \underline{v}_\ell \\ &= (\lambda_\ell - \lambda)\underline{v}_\ell\end{aligned}$$

Pick λ to increase the convergence rate:



Choose λ so that

$$\frac{\lambda_2 + \lambda_n}{2} \to 0$$

$$\Rightarrow \lambda = \frac{\lambda_2 + \lambda_n}{2}$$

<u>Idea Two</u> Apply the Power Method to the eigenvalues of A^{-1} (or better yet, $(A - sI)^{-1}$.

This is called <u>Inverse Power Method with Shift</u>.

If A has eigenvalues $\lambda_1, \dots, \lambda_n$, then A^{-1} has eigenvalues $\frac{1}{\lambda_1} \dots \frac{1}{\lambda_n}$, and $(A - sI)^{-1}$ has eigenvalues $\frac{1}{\lambda_1 - s}, \dots, \frac{1}{\lambda_n - s}$.

The proper choice of s can cause <u>very</u> rapid convergence.

Choosing s close to λ_ℓ causes $\frac{1}{\lambda_\ell - s}$ to grow very large, while $\frac{1}{\lambda_j - s}$, $j \neq \ell$, remains bounded.

This scheme is of course more expensive, since "applying" A^{-1} requires solving a linear system.

<u>Inverse Power Method with Shift</u>

1. set $\hat{\underline{b}}_0$ to be random.
2. solve $(A - sI)\,\underline{b}_1 = \hat{\underline{b}}_0$

$$\iff \underline{b}_1 = (A - sI)^{-1}\,\hat{\underline{b}}_0$$

3. set $\hat{\underline{b}}_1 = \frac{\underline{b}_1}{\|\underline{b}_1\|}$.

4. Proceed as in the Power Method...