Lecture 23 — Numerical Analysis — April 24, 2018

There are two types of ODEs:

Initial value problems (IVPs):

$$(*) \qquad \begin{aligned} y'(t) &= f(y,t) \\ y(t_0) &= y_0 \end{aligned}$$

and boundary value problems (BVPs):

Ex: $$ay'' + by' + cy = 0$$
$$y(0) = y_0, \quad y(1) = y_1$$

The numerical methods for their solution are different, and we will focus on IVPs.

The exact solution to $(*)$ is given by

$$y(t) = y_0 + \int_{t_0}^{t} f(y(\tau),\tau)\, d\tau$$

All numerical methods, in some form or another, approximate this exact solution.

The local uniqueness of the solution to (*) is established by Picard's Theorem, see page 311 in the textbook. Basically, if f is continuous and bounded in a neighborhood of (t_0, y_0), then a unique solution exists in some other neighborhood of (t_0, y_0).

Another way of viewing numerical solutions to (*) is to directly approximate the differential part:

$$y'(t) \approx \underbrace{\frac{y(t+h) - y(t)}{h}}_{\text{finite difference}} \approx f(y,t).$$

$\Rightarrow$ Given $y(t)$, to approximate $y(t+h)$ we write

$$\frac{y(t+h) - y(t)}{h} \approx f(y(t), t)$$

$$\Rightarrow \quad y(t+h) \approx y(t) + \underbrace{h\, f(y(t), t)}_{\approx \int_t^{t+h} f(y(t), t)\, dt}$$

This scheme is called the Forward Euler Method.

Let's turn to Finite Differences for a bit: approximation error and round-off error.

How good is the Forward Difference at approximating $y'(t)$ in the presence of round-off error?

round off

$$\frac{y(t+h) - y(t)}{h} \xrightarrow{f.p.} \frac{y(t+h)(1+\delta_1) - y(t)(1+\delta_2)}{h}$$

$(\delta < \epsilon,$ machine precision$)$

$$= \frac{y(t+h) - y(t)}{h} + \frac{y(t+h)\delta_1 + y(t)\delta_2}{h}$$

$$= \frac{\left(y(t) + h\,y'(t) + \frac{h^2}{2}y''(\xi)\right) - y(t)}{h} + \frac{y(t+h)\delta_1 + y(t)\delta_2}{h}$$

$$= y'(t) + \frac{h}{2}y''(\xi) + \frac{y(t+h)\delta_1 + y(t)\delta_2}{h}$$

The error is given as $Err(h) = y' - \dfrac{y(t+h) - y(t)}{h}$

$$\Rightarrow |Err| \le \left|\frac{h}{2} y''(\xi)\right| + \left|\frac{y(t+h)\,\delta_1 + y(t)\,\delta_2}{h}\right|$$

$$\le O(h) + \Big(|y(t+h)| + |y(t)|\Big)\frac{\epsilon}{h}$$

$$\le \underbrace{O(h)}_{\substack{\text{truncation}\\ \text{error}\\ \text{or approximation}\\ \text{error}}} + \underbrace{O\left(\frac{\epsilon}{h}\right)}_{\substack{\text{round-off}\\ \text{error}}}$$

To minimize this error, h must be chosen to balance these terms:

$$h \sim \frac{\epsilon}{h} \Rightarrow h \sim \sqrt{\epsilon}$$

If $\epsilon \sim 10^{-16}$, then choosing $h \sim 10^{-8}$ minimizes the error, which is then of size 10^{-8}.

That is the <u>best</u> you can hope for from a forward difference approximation.

An alternative approximation, the <u>centered difference</u>, has the property that

$$\left| y'(t) - \frac{y(t+h) - y(t-h)}{2h} \right| \le \underbrace{O(h^2)}_{\text{truncation error}} + \underbrace{O\left(\frac{\epsilon}{h}\right)}_{\text{round off error}}$$

$\Rightarrow$ choose $h^2 \sim \frac{\epsilon}{h}$ to balance terms

$\Rightarrow$ $h \sim \sqrt[3]{\epsilon}$ to minimize

With $h \sim 10^{-5}$, the total error above is $\sim 10^{-10}$.

This is a 2^{nd}-order accurate approximation.