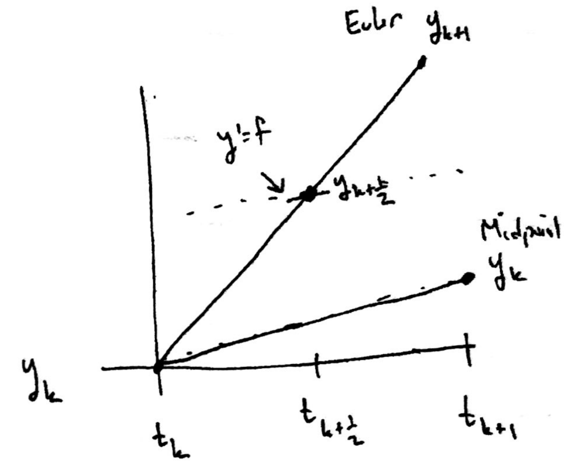
Mention CAS
Course Evals.

Midpoint method

Half-step, then full-step

$$y_{k+\frac{1}{2}} = y_k + \frac{h}{2} f(t_k, y_k)$$

$$y_k = y_k + h f(t_{k+\frac{1}{2}}, y_{k+\frac{1}{2}})$$



Compare with two half-Euler steps:

$$y_{k+\frac{1}{2}} = y_k + \frac{h}{2} f(t_k, y_k)$$

$$y_{k+1} = y_{k+1/2} + \frac{h}{2} f(t_{k+\frac{1}{2}}, y_{k+\frac{1}{2}})$$

$$= y_k + \frac{h}{2} f(t_k, y_k) + \frac{h}{2} f(t_{k+\frac{1}{2}}, y_{k+\frac{1}{2}})$$

$$= y_k + h\left(\frac{f(t_k, y_k) + f(t_{k+\frac{1}{2}}, y_{k+\frac{1}{2}})}{2}\right)$$ not the same rule.

Local truncation Error: $O(h^2)$

Quadrature method rules:

~~one-step version~~

$$y'(t) = f(t, y(t))$$

$$y(t_0) = y_0$$

One-step version:

$$y(t+h) = y(t) + \underbrace{\int_t^{t+h} f(\tau, y(\tau))\, d\tau}_{\text{use any quadrature rule to approximate}}$$

Solution: $y(t) = y(t_0) + \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau$

Trapezoidal Method

$$y(t+h) = y(t) + \int_t^{t+h} f(s, y(s))\, ds$$

$$\approx y(t) + \frac{h}{2}\Big(f(t, y(t)) + f(t+h, y(t+h)) \Big)$$

$$\Rightarrow \quad y_{k+1} = y_k + \frac{h}{2}\Big(f(t_k, y_k) + f(t_{k+1}, y_{k+1}) \Big)$$

need to solve for y_{k+1} $\Rightarrow$ Implicit method.

$\Rightarrow$ Euler, Midpoint were explicit methods

To avoid solving for y_{k+1}: approximate the Trap. Meth. with

Heun's method: ① Take Euler step to get $\tilde{y}_{k+1}$

② Use $\tilde{y}_{k+1}$ to estimate RHS:

$$\tilde{y}_{k+1} = y_k + h f(t_k, y_k)$$

$$y_{k+1} = y_k + \underbrace{\frac{h}{2}\Big(f(t_k, y_k) + f(t_{k+1}, \tilde{y}_{k+1}) \Big)}$$

average slope of curve through t_k, y_k and $t_{k+1}, \tilde{y}_{k+1}$

These methods can be extended: Runge-Kutta methods (next time)

Analysis of One-step methods

All explicit one-step methods are of the form:

$$y_{k+1} + y_k + \underbrace{h\psi(t_k, y_k, h)}_{\text{varies by how we estimate some derivative}}$$

Example: Midpoint method:

$$y_{k+\frac{1}{2}} = y_k + \frac{h}{2} f(t_k, y_k)$$

$$y_{k+1} = y_k + h f(t_{k+\frac{1}{2}}, y_{k+\frac{1}{2}})$$

$$\Rightarrow y_{k+1} = y_k + h f\left(t_{k+\frac{1}{2}}, y_k + \frac{h}{2} f(t_k, y_k)\right)$$

$$\Rightarrow \psi(t, y, h) = f\left(t + \frac{h}{2}, y + \frac{h}{2} f(t, y)\right)$$

Def: A one-step method is consistent if $\lim_{h \to 0} \psi(t, y, h) = f(t, y)$

$$\Leftrightarrow \lim_{h \to 0} \left(\frac{y(t_{k+1}) - y(t_k)}{h} - \psi(t_k, y(t_k), h) \right) = 0$$

Def: A one-step method is stable if there is some $K > 0$, $h_0 > 0$ such that two solutions $y_n, \tilde{y}_n$ have

$$|y_n - \tilde{y}_n| \leq K |y_0 - \hat{y}_0| \text{ whenever } h \leq h_0 \text{ and } \underset{\substack{\uparrow \\ \text{number of steps.}}}{n} h \leq T - t_0$$

Thm: If an explicit one-step method is stable and consistent and has local truncation error $\mathcal{O}(h^p)$, then the global error is $\mathcal{O}(h^p)$.

Def: One-step method is convergent if

$$\max_{t_k \in [t_0, T]} |y(t_k) - y_k| \to 0 \quad \text{as } y_0 \to y(t_0) \text{ and } h \to 0.$$

Take home: Stable + consistent $\Rightarrow$ convergent.

Another example (Stiff IVP).

$$y_1' = -100y_1 + y_2$$

$$y_2' = \frac{-y_2}{10}$$

$$\Rightarrow \quad \cancel{\&}\ \vec{y}' = A\vec{y} \qquad \left| \quad \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -100 & 1 \\ 0 & -1/10 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right.$$

$$\Rightarrow \quad y_2(t) = y_2(0)e^{-t/10}$$

$$\Rightarrow \quad y_1(t) = \underset{\substack{\uparrow \\ \text{decays extremely fast.}}}{c_1 e^{-100t}} + c_2 e^{-t/10}$$

How small ~~should~~ should the time step h be?

Try Euler's method:

2nd Eqn

$$y_{2,k+1} = y_{2,k} - \frac{h y_{2,k}}{10}$$

$$\Rightarrow \quad y_{2,k} = \left(1 - \frac{h}{10}\right)^k y_2(0)$$

1st Eqn

$$y_{1,k+1} = y_{1,k} + h \cdot (-100\, y_{1,k} + y_{2,k})$$

$$= (1 - 100h)\, y_{1,k} + \underset{\substack{\uparrow \\ (1-\frac{h}{10})^k y_2(0)}}{h y_{2,k}}$$

... continue back substituting...

$$y_{1,k+1} = (1-100h)^{k+1}\, y_1(0) \; + \; h\left(1-\frac{h}{10}\right)^{k}\left(\sum_{l=0}^{k}\left(\frac{1-100h}{1-h/10}\right)^{l}\right) y_2(0)$$

$$= c_1 \underbrace{(1-100h)}^{k+1} + c_2\left(1-\frac{h}{10}\right)^{k+1}$$

need $(1-100h) < 1$

$\Rightarrow h < \frac{1}{50}$

~~Even~~ This term will grow even once e^{-100t} has decayed.

leads to wild oscillations

$\Rightarrow$ stiff equations $\rightarrow$ have components that behave on different time-scales.