Homework 1

Due: 12:30pm February 8, 2018

Notes on the first (and all subsequent) assignments:

Submission: Homework assignments must be submitted in the class on the due date. If you cannot attend the class, please send your solution per email as a single PDF before class. Please hand in cleanly handwritten or typed (preferably with LaTeX) homework. Feel free to use original homework LaTeX document to write-up your homework. If you are required to hand in code or code listings, this will explicitly be stated on that homework assignment.

Collaboration: NYU's integrity policies will be enforced. You are encouraged to discuss the problems with other students in person (or on Piazza). However, you must write (i.e., type) every line of code yourself and also write up your solutions independently. Copying of any portion of someone else's solution/code or allowing others to copy your solution/code is considered cheating.

Plotting and formatting: Plot figures carefully and think about what you want to illustrate with a plot. Choose proper ranges and scales (semilogx, semilogy, loglog), always label axes, and give meaningful titles. Sometimes, using a table can be useful, but never submit pages of numbers. Discuss what we can observe in and learn from a plot. If you do print numbers, in MATLAB for example, use `fprintf` to format the output nicely. Use `format compact` and other format commands to control how MATLAB prints things. When you create figures using MATLAB (or Python/Octave), please try to export them in a vector graphics format (.eps, .pdf, .dxf) rather than raster graphics or bitmaps (.jpg, .png, .gif, .tif). Vector graphics-based plots avoid pixelation and thus look much cleaner.

Programming: This is an essential part of this class. We will use MATLAB for demonstration purposes in class, but you are free to use other languages. The TA will give an introduction to MATLAB in the first few recitation classes. Please use meaningful variable names, try to write clean, concise and easy-to-read code and use comments for explanation.

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1. **[10pts]** In this problem, you will write a code that will find all of the roots of the function $f(x) = x^5 - 3x^2 + 1$ inside the interval $[-2, 2]$. For each part, please submit a print-out of your code along with the values you obtain for the roots.

   (a) Using the fact that the roots of f on this interval are separated by at least 0.25, write a program that implements the bisection algorithm to obtain estimates of the roots that are accurate to within 0.1.

   (b) Using the estimates of the roots obtained in the previous part, write a program that implements Newton's method to refine the values of the roots such that the difference between successive iterates is at most 10^{-12}.

2. **[10pts]** Let the function f be twice continuously differentiable in a neighborhood of ξ (meaning that f, f', and f'' are all continuous in a neighborhood of ξ). Furthermore, let $f(\xi) = f'(\xi) = 0$ and $f''(\xi) > 0$. The function f is said to have a double root.

   (a) At what rate do you expect Newton' method to converge?

   (b) Write a program to verify your estimated convergence rate for the function $f(x) = x^3$, starting with the initial guess $x_0 = -0.5$. Turn in your code, as well as a table of values output by Newton's method.

(c) Re-run your program, except this time on the function $f(x) = x^3 - 1$. Use the starting value $x_0 = 0.5$. Turn in your code, as well as a table of values output by Newton's method.

3. **[10pts]** In this problem, you will prove the rate of convergence for the secant method.

(a) Show that the secant method

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k)$$

can be rewritten in the form:

$$x_{k+1} = \frac{x_k f(x_{k-1}) - x_{k-1} f(x_k)}{f(x_{k-1}) - f(x_k)}. \tag{1}$$

(b) Now, denote the root of f to be ξ, so that $f(\xi) = 0$. Also assume that f is twice continuously differentiable and that $f' > 0$ and $f'' > 0$ in a neighborhood of ξ. Define the quantity φ to be:

$$\varphi(x_k, x_{k-1}) = \frac{x_{k+1} - \xi}{(x_k - \xi)(x_{k-1} - \xi)},$$

where x_{k+1} is as in (1). Compute (for fixed value of x_{k-1})

$$\psi(x_{k-1}) = \lim_{x_k \to \xi} \varphi(x_k, x_{k-1}).$$

(c) Now compute $\lim_{x_{k-1} \to \xi} \psi(x_{k-1})$, and therefore show that

$$\lim_{x_k, x_{k-1} \to \xi} \varphi(x_k, x_{k-1}) = \frac{f''(\xi)}{2f'(\xi)}.$$

(d) Next, assume that the secant method has convergence order q, that is to say that

$$\lim_{k \to \infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|^q} = A < \infty.$$

Using the above results, show that $q - 1 - 1/q = 0$, and therefore that $q = (1 + \sqrt{5})/2$.

(e) Finally, show that this implies that

$$\lim_{k \to \infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|^q} = \left(\frac{f''(\xi)}{2f'(\xi)} \right)^{q/(1+q)}.$$

4. **[10pts]** Let the function f be continuously differentiable on the interval $[a, b]$. Furthermore, let its derivative f' satisfy $f' < 1$. Does this imply that f is a contraction on $[a, b]$? If so, prove it.

5. **[10pts]** The behavior of fixed point iterations lead to some of the first examples encountered when studying chaotic systems. Consider the *tent map* on the interval $[0, 1]$:

$$f(x) = a(0.5 - |x - 0.5|), \text{with } a \in (0, 2].$$

(a) How many fixed points does f have on the interval $[0, 1]$? What are they (in terms of a)?

(b) Are they stable? Unstable?

(c) For $a = 2$, make a table of the first 10 iterates $x_{k+1} = f(x_k)$ for two starting values: $x_0 = 0.399$ and $x_0 = 0.400$. What is happening?

(d) Can you explain the behavior of this iteration using any of the techniques learned in class or in the textbook?