# Homework 2

Due: 12:30pm February 22, 2018

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Notes on the assignment:

**Submission**: (New procedure.) Homework assignments must be submitted **via email** to the following address:

`hw2.rt0e5c61owt6ogqx@u.box.com`

by the start of class on the due date. Homework submitted after this time will **not be accepted**. Your submission should take the form of a *single file* with the filename `yourNetID_hw2.zip` attached to an email to the above address. The subject line and body of the email will be ignored, so please do not include any other comments aside from the attached file. For example, if your NYU netID is `abc123`, then you should submit a single file with filename `abc123_hw2.zip`. The `.zip` archive should include a PDF of the written portion of your homework, and any files required for the programming aspect of your homework. Please prepare cleanly handwritten or typed (preferably with LaTeX) homework, and make sure that your name is on the homework. Feel free to use original homework LaTeX document to write-up your homework. If you are required to hand in code, this will explicitly be stated on that homework assignment.

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1. Newton's method can be extended to *matrix-functions* as well. For example, given a square matrix $\boldsymbol{A}$ and real number $t$, the *matrix-exponential* $e^{t\boldsymbol{A}}$ is defined via the Taylor series for the exponential function:

$$e^{t\boldsymbol{A}} = \boldsymbol{I} + t\boldsymbol{A} + \frac{(t\boldsymbol{A})^2}{2!} + \frac{(t\boldsymbol{A})^3}{3!} + + \frac{(t\boldsymbol{A})^4}{4!} + \dots \tag{1}$$

Obviously, the matrix $\boldsymbol{A}$ must be square.

(a) **[4pts]** Derive Newton's method for finding the root of an arbitrary matrix-valued function $f = f(\boldsymbol{X})$, where by *root* we mean that $\boldsymbol{X}$ is a root of $f$ if $f(\boldsymbol{X}) = \boldsymbol{0}$, where $\boldsymbol{0}$ is the matrix of all zeros. Assume that the matrix arguments of $f$ are square and invertible.

(b) **[3pts]** The square root of a matrix $\boldsymbol{A}$ is a matrix $\boldsymbol{X}$ such that $\boldsymbol{X}^2 = \boldsymbol{A}$. For a symmetric positive-definite matrix $\boldsymbol{A}$, derive the Newton iteration for finding $\boldsymbol{X} = \sqrt{\boldsymbol{A}}$.

(c) **[3pts]** Write a program using the Newton iteration that you derived above to find the square root of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 8 & 4 & 2 & 1 \\ 4 & 8 & 4 & 2 \\ 2 & 4 & 8 & 4 \\ 1 & 2 & 4 & 8 \end{pmatrix}. \tag{2}$$

The stopping criterion for your Newton iteration should be when the absolute difference between elements of successive iterations is at most $10^{-10}$. Submit your code for this part of the problem. Your code must be easily executable, submit instructions if necessary.

2. **[10pts]** For a yearly interest rate $0 < r < 1$ compounded over $n$ intervals, an amount of money $C$ grows to be

$$f(C, r, n) = C\left(1 + \frac{r}{n}\right)^n \tag{3}$$

after one year.

Let $r = 0.025$. If $n$ is *extremely* large, say $n = 10^{16}$, in IEEE double precision arithmetic,

$$f\left(1.0, 0.025, 10^{16}\right) = 1.0, \tag{4}$$

when in fact,

$$\begin{aligned} f(1.0, 0.025, 10^{16}) &\approx \lim_{n\to\infty} \left(1 + \frac{0.025}{n}\right)^n \\ &= e^{0.025} \\ &\approx 1.025315\ldots \end{aligned} \tag{5}$$

What happened? Without using the above approximation, is there another way to evaluate $f$ and not lose these digits of accuracy? (Assume you are only allowed to use double precision accuracy.)

3. Let the matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, with $m \geq n$, have rank $n$ and be such that when computing its $\boldsymbol{LU}$ decomposition $\boldsymbol{A} = \boldsymbol{LU}$ no pivoting whatsoever is needed.

   (a) **[5pts]** Carefully derive the computational cost (in floating point operations, referred to as flops) of computing $\boldsymbol{A} = \boldsymbol{LU}$. Assume that addition, subtraction, multiplication, and division each count as 1 flop. For example, computing $x = a(b+c)/d - e$ requires 4 flops.

   (b) **[5pts]** Now suppose that the matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is such that in computing its $\boldsymbol{LU}$ factorization row-pivoting is required. This means that we can write $\boldsymbol{PA} = \boldsymbol{LU}$ where $\boldsymbol{P}$ is a permutation matrix. Find $\boldsymbol{P}$, $\boldsymbol{L}$, and $\boldsymbol{U}$ for the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 6 & 6 & 0 \\ 3 & 0 & 2 \end{pmatrix}. \tag{6}$$

   If you do not show your work, you will not receive any credit for the problem.

4. (a) **[5pts]** Let $\boldsymbol{v} = (v_1, \ldots, v_n)^T \in \mathbb{R}^n$. Prove the following two inequalities:

$$\begin{aligned} \|\boldsymbol{v}\|_\infty &\leq \|\boldsymbol{v}\|_2, \\ \|\boldsymbol{v}\|_2^2 &\leq \|\boldsymbol{v}\|_1 \|\boldsymbol{v}\|_\infty. \end{aligned} \tag{7}$$

   For each inequality, give an example of a non-zero vector $\boldsymbol{v}$ for which equality is obtained. Finally, show that $\|\boldsymbol{v}\|_\infty \leq \|\boldsymbol{v}\|_2 \leq \|\boldsymbol{v}\|_1$ and that $\|\boldsymbol{v}\|_2 \leq \sqrt{n}\|\boldsymbol{v}\|_\infty$.

   (b) **[5pts]** Next, let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ be any real-valued matrix. Show that

$$\|\boldsymbol{A}\|_\infty \leq \sqrt{n}\|\boldsymbol{A}\|_2 \qquad \text{and} \qquad \|\boldsymbol{A}\|_2 \leq \sqrt{m}\|\boldsymbol{A}\|_\infty. \tag{8}$$

   For each inequality, give an example matrix $\boldsymbol{A}$ for which equality is attained.