Homework 5

Due: 12:30pm April 19, 2018

Notes on the assignment:

Submission: Homework assignments must be submitted in hardcopy at the beginning of class on the due date. Late homework will not be accepted. You are encouraged to use the original homework LaTeX document as a template to write-up your homework. If you are required to hand in code, this will explicitly be stated on the homework assignment.

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1. (10 pts) When solving an equation of the form $f(x) = 0$, Newton's method uses explicit derivative information of f to calculate the tangent line and its x-intercept (its root). The secant method uses two points $(x_{k-1}, f(x_{k-1}))$, and $(x_k, f(x_k))$, to fit a linear function approximating the tangent line and then finds its root.

   On the other hand, Muller's method uses three points, $(x_{k-2}, f(x_{k-2}))$, $(x_{k-1}, f(x_{k-1}))$, $(x_k, f(x_k))$, to fit a parabola (i.e. interpolate a parabola). It then uses the quadratic formula to find the root of this parabola that is closest to x_k. Derive Muller's method. To obtain full credit, you must show your work.

2. (a) (3 pts) Write down the interpolating polynomial in Lagrange form of degree 1 for the function $f(x) = x^3$ using the points $x_0 = 0$ and $x_1 = a$.

   (b) (3 pts) Theorem 6.2 in the text states that for $x \in [0, a]$, there exists a $\xi = \xi(x) \in (0, a)$ such that
   $$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j).$$
   Here, n is the degree of the interpolating polynomial p_n. Verify the above formula, by direct calculation, for the function and interpolating polynomial from part (a). Show that in this case, ξ is unique and has the value $\xi = \frac{1}{3}(x + a)$.

   (c) (4 pts) Repeat parts (a) and (b) for the function $f(x) = (2x - a)^4$. This time, show that there are two possible values for ξ, and give their values.

3. (10 pts) In class, we provided two equivalent formulas defining Chebyshev polynomials:
   $$T_n(x) = \cos(n \arccos x),$$
   and
   $$T_0(x) = 1,$$
   $$T_1(x) = x,$$
   $$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$
   Show (prove) that these formulas are equivalent.

   Hint: Use formulas for the cosine of a sum and difference of two angles to show that
   $$\cos((n+1) \arccos x) = 2x \cos(n \arccos x) - \cos((n-1) \arccos x).$$

4. (a) (5 pts) Construct the orthogonal polynomials of degrees 0, 1, and 2 on the interval $(0, 1)$ with the weight function $w(x) = -\log x$. (Note that by $\log x$ we mean that $e^{\log x} = x$.)

   (b) (5 pts) Now, assume that the weight function $w(x) = 1$. The function H is defined by:

$$H(x) = \begin{cases} 1 & x > 0 \\ -1 & x \leq 0. \end{cases}$$

   Construct the best polynomial approximation of degrees 0, 1, and 2 in the 2-norm to this function over the interval (-1,1). The function H is discontinuous, but still square integrable and therefore has a unique, convergent decomposition in orthogonal polynomials on this interval.